Solutions to Exercises Set 1

1. i When $a = 10$, $n = f(10) = 60(10) - 900 = -300$. This represents a negative number of words for a 10 month old baby so the formula does not make sense for $a = 10$.

ii You would not expect an adult to learn new words at the same rate as that of a very young child.

iii You can calculate the rate at which the size of the child's vocabulary is changing by looking at the graph drawn on page 4. If you take two points on the graph say when $x = 30$ and $x = 40$, the corresponding values for $f(x)$ are 900 and 1500 respectively. We can determine the rate the child's vocabulary is changing by calculating,

$$\text{rate} = \frac{\text{change in } f(x)}{\text{change in } x} = \frac{1500 - 900}{40 - 30} = \frac{600}{10} = 60 \quad \text{words per month.}$$

2. i When $t = 3$,

$$d = f(3) = 4.9(3)^2 = 44.1.$$

Therefore the object falls 44.1 metres in the first three seconds.

When $t = 6$,

$$d = f(6) = 4.9(6)^2 = 176.4.$$

The object falls 176.4 metres in the first six seconds.

Therefore the object falls $176.4 - 44.1 = 132.3$ metres in the second three seconds.

ii The speed of the falling object is increasing, as it falls 44.1 metres in the first three seconds compared with 132.3 metres in the next three seconds.

iii We can calculate the average speed of the object as:

$$\text{average speed} = \frac{\text{distance fallen}}{\text{time taken}}.$$

Therfore the average speed of the object over the first 3 seconds is $\frac{44.1}{3} = 14.7$ m/sec while the average speed is $\frac{132.3}{3} = 44.1$ m/sec over the next 3 seconds.

3. i If $f(x) = 5(x - 3)$,

$$f(-6) = 5((-6) - 3) = 5(-6 - 3) = 5(-9) = -45.$$

(Substitute -6 everywhere there is an x.)

$$f(a) = 5((a) - 3) = 5(a - 3).$$

ii If $f(x) = t^3 - 5t^2$,

$$f(-2) = (-2)^3 - 5(-2)^2 = -8 - 5(4) = -28.$$

$$f(\frac{1}{2}) = (\frac{1}{2})^3 - 5(\frac{1}{2})^2 = \frac{1}{8} - 5(\frac{1}{4}) = \frac{1}{8} - \frac{5}{4} = -\frac{9}{8}.$$

iii If $f(x) = 4x - 2$, $f(x) = 7$ when

$$\begin{aligned} 4x - 2 &= 7 \\ 4x &= 9 \\ x &= \frac{9}{4}. \end{aligned}$$

Solutions to Exercises Set 2

1. i **c.**

$$\begin{aligned} 3x - 2y + 6 &= 0 \\ 3x + 6 &= 2y \\ y &= \frac{3x+6}{2} \\ &= \frac{3}{2}x + 3. \end{aligned}$$

f.

$$\begin{aligned} y + 1 &= 3(2x - 1) \\ y &= 3(2x-1) - 1 \\ &= 6x - 3 - 1 \\ &= 6x - 4. \end{aligned}$$

2. i **a.** The general equation of the line is $y = mx + b$ where m is the gradient of the line. Therefore the equation must be of the form $y = 2x + b$.
Since the line passes through the point $(4, 1)$, when $x = 4$, $y = 1$ so we have $1 = 2(4) + b$, and hence $b = -7$.
The equation of the line has the form $y = 2x - 7$.

b. The equation of the line has the form $y = \frac{1}{3}x + b$.
When $x = 1$, $y = -3$ so we have $-3 = \frac{1}{3}(1) + b$, and hence $b = -\frac{10}{3}$.
The equation of the line is $y = \frac{1}{3}x - \frac{10}{3}$ or $3y = x - 10$.

c. The equation of the line must have the form $y = -3x + b$.
When $x = 2$, $y = 1$ so we have $1 = -3(2) + b$, and hence $b = 7$.
The equation of the line is $y = -3x + 7$.

d. The equation of the line has the form $y = -1x + b = -x + b$.
When $x = p$, $y = q$ so $q = -(p) + b$, and hence $b = p + q$.
The equation of the line is $y = -x + p + q$.

ii The answers are given in the notes. We've worked out the first one for you.

a. When $x = 0$, $y = 2(0) - 7 = -7$ so the y-intercept is -7.
When $y = 0$, $0 = 2x - 7$ ie $x = \frac{7}{2}$ so the x-intercept is $\frac{7}{2}$.

3. i The gradient of the line given by

$$m = \frac{\text{change in } y}{\text{change in } x}.$$

a. $m = \frac{6-3}{4-2} = \frac{3}{2}$.

b. $m = \frac{-2-1}{1-(-2)} = \frac{-3}{3} = -1$.
Notice the use of the bracket in the denominator $1-(-2)$, which helps us get the sign correct.

c. $m = \frac{1-2}{3-(-1)} = \frac{-1}{4} = -\frac{1}{4}$.

d. $m = \frac{-2-(-2)}{-3-4} = 0$.

e. $m = \frac{-2-0}{0-3} = \frac{-2}{-3} = \frac{2}{3}$.

ii In each of the following examples we use the gradient m we found in the previous exercise in the equation $y = mx + b$ and use one of the given points to find b by substitution.

a. $m = \frac{3}{2}$ so the equation of the line is $y = \frac{3}{2}x + b$.

We can use either the point $(2, 3)$ or the point $(4, 6)$ to find b.

When $x = 2$, $y = 3$ so $3 = \frac{3}{2}(2) + b$. Therefore $b = 0$.

So the equation of the line is $y = \frac{3}{2}x$ or $3x - 2y = 0$.

b. The equation of the line is $y = -x + b$.

When $x = -2$, $y = 1$ so $1 = -(-2) + b$ ie $b = -1$.
(Note the use of the bracket when we substituted)

The equation of the line is $y = -x - 1$ or $x + y + 1 = 0$.

c. The equation of the line is $y = -\frac{1}{4}x + b$.

When $x = -1$, $y = 2$ so $2 = -\frac{1}{4}(-1) + b$, ie $b = \frac{7}{4}$.

The equation of the line is $y = -\frac{1}{4}x + \frac{7}{4}$ or $x + 4y - 7 = 0$.

d. The equation of the line is $y = 0x + b$.

When $x = 4$, $y = -2$ so $-2 = b$.

The equation of the line is $y = -2$ or $y + 2 = 0$.

e. The equation of the line is $y = \frac{2}{3}x + b$.

When $x = 3$, $y = 0$ so $0 = \frac{2}{3}(3) + b$, ie $b = -2$.

The equation of the line is $y = \frac{2}{3}x - 2$ or $2x - 3y - 6 = 0$.

iii The answers to these questions are given in the notes. We've worked out the first two here to show you how.

a. x-intercept:
When $y = 0$, $3x - 2y = 0$ becomes $3x = 0$ so the x-intercept is 0.

y-intercept:
When $x = 0$, $3x - 2y = 0$ becomes $-2y = 0$ so the y-intercept is 0.

b. x-intercept:
When $y = 0$, $x+y+1 = 0$ becomes $x+0+1 = 0$ ie $x = -1$. So the x-intercept is -1.

y-intercept:

When $x = 0$, $y + 1 = 0$ so $y = -1$, The y-intercept is -1.

4. The line through $(4, 7)$ parallel to $y = 3x + 7$ must have the same gradient ie $m = 3$.

Therefore the line has equation $y = 3x + b$.

When $x = 4$, $y = 7$ so $7 = 3(4) + b$, ie $b = -5$.

The equation of the line is $y = 3x - 5$ or $3x - y - 5 = 0$.

5. First of all we need to find the equation of the line.

The gradient of the line through the points $(2, 5)$ and $(5, 9)$ is given by $m = \frac{9-5}{5-2} = \frac{4}{3}$.

The equation of the line is $y = \frac{4}{3}x+b$. Substituting $x = 2$ and $y = 5$ gives $5 = \frac{4}{3}(2)+b$ so $3b = 15 - 8 = 7$ ie $b = \frac{7}{3}$.

The equation of the line is $y = \frac{4}{3}x + \frac{7}{3}$ or $4x - 3y + 7 = 0$.

The line we've found intersects the vertical line $x = 7$ when

$$4(7) - 3y + 7 = 0 \quad \text{ie} \quad y = \frac{35}{3}.$$

Therefore the line through $(2, 5)$ and $(5, 9)$ intersects the vertical line $x = 7$ at $(7, \frac{35}{3})$.

Solutions to Selected Exercises Set 3

1. The answers are given in the notes.

2. **i** **a.** $(x+3)(x+3) = x(x+3) + 3(x+3) = x^2 + 3x + 3x + 9 = x^2 + 6x + 9.$

c. $(x+5)(x+5) = x^2 + 5x + 5x + 25 = x^2 + 10x + 25.$

f. $(x+a)(x+a) = x^2 + ax + ax + a^2 = x^2 + 2ax + a^2.$
This one gives us the general pattern.

$$(x+a)^2 = (x+a)(x+a) = x^2 + 2ax + a^2.$$

The $2ax$ is sometimes called the cross term.

ii **a.** $(x-3)(x-3) = x(x-3) - 3(x-3) = x^2 - 3x - 3x + 9 = x^2 - 6x + 9.$

d. $(x-4)^2 = (x-4)(x-4) = x^2 - 4x - 4x + 16 = x^2 - 8x + 16.$

f. $(x-a)^2 = (x-a)(x-a) = x^2 - ax - ax + a^2 = x^2 - 2ax + a^2.$
This one gives us the general pattern.

iii **a.** $(x+3)(x-3) = x(x-3) + 3(x-3) = x^2 - 3x + 3x - 9 = x^2 - 9.$

d. $(x-a)(x+a) = x(x+a) - a(x+a) = x^2 + ax - ax - a^2 = x^2 - a^2.$

This one gives us the pattern. It is often written as

$$x^2 - a^2 = (x+a)(x-a)$$

and is referred to as the difference of two squares.

iv **a.** $(x+3)(x+2) = x(x+2) + 3(x+2) = x^2 + 2x + 3x + 6 = x^2 + 5x + 6.$

b. $(x+3)(x-2) = x(x-2) + 3(x-2) = x^2 - 2x + 3x - 6 = x^2 + x - 6.$

c. $(x-3)(x+2) = x(x+2) - 3(x+2) = x^2 + 2x - 3x - 6 = x^2 - x - 6.$

d. $(x-3)(x-2) = x(x-2) - 3(x-2) = x^2 - 2x - 3x + 6 = x^2 - 5x + 6.$

The pattern exhibited by these examples can be summed up as follows:

a. and **d.**
If the signs in the brackets are the same then the constant is positive as the constant is the product of the constants in the brackets. Also, the coefficient of the cross term is sum of the constants in the brackets and has the same sign as them.

b. and **c.**
If the signs in the brackets are different then the constant is negative as the constant is the product of the constants in the brackets. Also, the coefficient of the cross term is the difference of the constants in the brackets and care must be taken to assign the correct sign to each one.

v These are very similar to the previous examples.

vi **a.** $(2x + 1)(x + 3) = 2x(x + 3) + 1(x + 3) = 2x^2 + 6x + x + 3 = 2x^2 + 7x + 3.$

b. $(2x + 1)(x - 3) = 2x^2 - 6x + x - 3 = 2x^2 - 5x - 3.$

c. $(2x - 1)(x + 3) = 2x(x + 3) - 1(x + 3) = 2x^2 + 6x - x - 3 = 2x^2 + 5x - 3.$

d. $(2x - 1)(x - 3) = 2x^2 - 6x - x + 3 = 2x^2 - 7x + 3.$

In the previous exercises we needed to take into account the coefficient of x^2 when we worked out the cross term.

3. We now need to use the patterns we spotted in the previous exercise to factorise the following exercises. We will some of each type.

a. $x^2 + 4x + 3$

The $+3$ tells us that the signs in the brackets are the same, while the $+4x$ tells us that the sign must be $+$.

This gives us $(x + \cdot)(x + \cdot)$.

The product of the constants in the brackets equals 3, while their sum equals 4. Therefore 3 and 1 will do.

We get

$$x^2 + 4x + 3 = (x + 3)(x + 1), \quad \text{which we check by expanding again.}$$

d. $x^2 + 7x + 12$

$+12$ tells us that the signs are the same and $+7x$ tells us that they are $+$. The constants in the brackets must multiply to 12 and add to 7, so must be 6 and 1.

$$x^2 + 7x + 12 = (x + 6)(x + 1).$$

f. $x^2 - 3x - 10$

The -10 tell us that the signs in the brackets are different so we are looking for factors of -10 with a difference of -3. They must be -5 and 2.

$$x^2 - 3x - 10 = (x - 5)(x + 2) \quad \text{again check by expanding.}$$

(Notice if we chose 5 and -2 we would get $x^2 + 3x - 10$.)

j. $x^2 - x - 12$

-12 tells us that the signs in the brackets are different and we are looking for factors of -12 with a difference of -1. Take -4 and 3.

$$x^2 - x - 12 = (x - 4)(x + 3).$$

k. $x^2 - 11x + 10$

$+10$ tells us that the signs in the brackets are the same and $-11x$ tells us the signs are $-$. We are looking for factors of 10 whose sum is 11, so take 10 and 1.

$$x^2 - 11x + 10 = (x - 10)(x - 1).$$

m. $x^2 - 16x + 15$

$+15$ tells us that the signs in the brackets are the same and $-16x$ tells us that they are $-$. We are looking for factors of 15 whose sum is 16. Take 15 and 1.

$$x^2 - 16x + 15 = (x - 15)(x - 1).$$

s. $x^2 + 5x - 14$

We are looking for factors of -14 whose difference is 5, so take 7 and -2.

$$x^2 + 5x - 14 = (x + 7)(x - 2).$$

v. $m^2 + 6m + 9$

We are looking for factors of 9 with a sum of 6. Take 3 and 3.

$$m^2 + 6m + 9 = (m + 3)(m + 3) = (m + 3)^2.$$

w. $2x^2 + 7x + 3$

We know that the signs in the brackets are the same and must be $+$.

So we have $(2x + \cdot)(x + \cdot)$ as the only factors of 2 are 2 and 1.

We are looking for factors of 3 so try 3 and 1. Now try them in the bracket to see if we can get it to work.

$$(2x + 3)(x + 1) = 2x^2 + 5x + 3 \quad \text{which is not what we want.}$$

If we swop them over we get

$$(2x + 1)(x + 3) = x^2 + 7x + 3 \quad \text{which is what we were after.}$$

There is an element of trial and error in doing these exercises!

4. **a.**

$$\begin{aligned} x^2 - 7x + 12 &= 0 \\ (x - 6)(x - 1) &= 0 \end{aligned}$$

Therefore $x - 6 = 0$ or $x - 1 = 0$. That is, $x = 6$ or $x = 1$.

b.

$$\begin{aligned} x^2+3x &= 0 \\ x(x+3) &= 0 \end{aligned}$$

So, $x = 0$ or $x + 3 = 0$. That is, $x = 0$ or $x = -3$.

d.

$$\begin{aligned} 4x^2-4x &= 0 \\ 4x(x-1) &= 0 \end{aligned}$$

So, $x = 0$ or $x = 1$.

e.

$$\begin{aligned} x^2-4x-5 &= 0 \\ (x-5)(x+1) &= 0 \end{aligned}$$

So, $x = 5$ or $x = -1$.

h.

$$\begin{aligned} 5x-x^2 &= 0 \\ x(5-x) &= 0 \end{aligned}$$

So, $x = 0$ or $5 - x = 0$, ie $x = 5$.

k.

$$\begin{aligned} x^2-81 &= 0 \\ (x+9)(x-9) &= 0 \end{aligned}$$

So, $x = -9$ or $x = +9$. This is the difference of two squares.

m.

$$\begin{aligned} x^2-1 &= 0 \\ (x+1)(x-1) &= 0 \end{aligned}$$

So, $x = -1$ or $x = +1$.

n.

$$\begin{aligned} 9x^2 - 16 &= 0 \\ (3x)^2 - 16 &= 0 \\ (3x+4)(3x-4) &= 0 \end{aligned}$$

So, $x = -\frac{4}{3}$ or $x = \frac{4}{3}$.

o.

$$\begin{aligned} x^2 - 2x + 1 &= 0 \\ (x-1)(x-1) &= 0 \end{aligned}$$

So, $x = 1$. Notice both brackets give the same answer.

r.

$$\begin{aligned} x^2 - 14x + 49 &= 0 \\ (x-7)(x-7) &= 0 \end{aligned}$$

So, $x = 7$.

s.

$$\begin{aligned} 10 + 3x - x^2 &= 0 \\ x^2 - 3x - 10 &= 0 \\ (x-5)(x+2) &= 0 \end{aligned}$$

So, $x = 5$ or $x = -2$. Notice we started by rearranging the equation to make the x^2 term positive.

u.

$$\begin{aligned} 2x^2 - 7x + 5 &= 0 \\ (2x-5)(x-1) &= 0 \end{aligned}$$

So, $x = \frac{5}{2}$ or $x = 1$.

v.

$$\begin{aligned} 6x^2 - 25x - 9 &= 0 \\ (2x-9)(3x+1) &= 0 \end{aligned}$$

So, $x = \frac{9}{2}$ or $x = -\frac{1}{3}$. It may take you several goes to get this one out as there is more than one way to factorise both 6 and 9.

5. For these next exercises we will use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

a. $x^2 - 3x - 5 = 0$

$$\begin{aligned} x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-5)}}{2(1)} \\ &= \frac{3 \pm \sqrt{9 + 20}}{2} \\ &= \frac{3 \pm \sqrt{29}}{2} \end{aligned}$$

That is $x = \frac{3+\sqrt{29}}{2}$ or $x = \frac{3-\sqrt{29}}{2}$.

c. $x^2 + 6x + 2 = 0$

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{6^2 - 4(1)(2)}}{2(1)} \\ &= \frac{-6 \pm \sqrt{36 - 8}}{2} \\ &= \frac{-6 \pm \sqrt{28}}{2} \\ &= \frac{-6 \pm 2\sqrt{7}}{2} & \sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7} \\ &= -3 \pm \sqrt{7} \end{aligned}$$

That is, $x = -3 + \sqrt{7}$ or $x = -3 - \sqrt{7}$.

e. $3x^2 - x - 3 = 0$

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-3)}}{2(3)} \\ &= \frac{1 \pm \sqrt{1 + 36}}{6} \\ &= \frac{1 \pm \sqrt{37}}{6} \end{aligned}$$

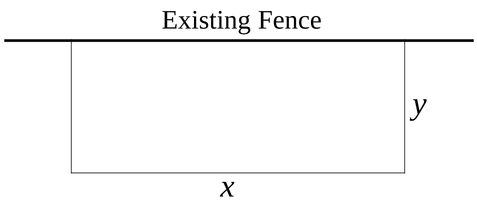
That is, $x = \frac{1+\sqrt{37}}{6}$ or $x = \frac{1-\sqrt{37}}{6}$.

f. $x^2 + 5x + 7 = 0$

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5^2 - 4(1)(7)}}{2(1)} \\ &= \frac{-5 \pm \sqrt{25-28}}{2} \\ &= \frac{-5 \pm \sqrt{-3}}{2} \end{aligned}$$

But here we have the $\sqrt{-3}$ which is not a real number, so there are no real solutions to this quadratic.

6. We approach these type of questions by drawing a picture and defining the variable we need.



Let x be the length of the side parallel to the existing fence and let y be the other side.

We have 300m of fencing material so $x + 2y = 300$. (1)

We want to fence an area of $10000m^2$, so $xy = 10000$. (2)

Now we need to eliminate either x or y to solve this equation.

Rearranging equation (1) we get $x = 300 - 2y$.

Sustituting in equation (2) we get $(300 - 2y)(y) = 10000$, ie $300y - 2y^2 = 10000$.

This gives us the quadratic $2y^2 - 300y + 10000 = 0$ which we can solve using the quadratic formula.

$$\begin{aligned} y &= \frac{300 \pm \sqrt{300^2 - 4(2)(10000)}}{2(2)} \\ &= \frac{300 \pm \sqrt{90000 - 80000}}{4} \\ &= \frac{300 \pm 100}{4} \end{aligned}$$

So, $y = \frac{300+100}{4} = 100$, or $y = \frac{300-100}{4} = 50$.

If $y = 100$, $x = 300 - 2(100) = 100$. If $y = 50$, $x = 300 - 2(50) = 200$.

So the dimensions of the rectangle are either 100m by 100m or 200m by 50m.

8. The rock will reach the ground when $s = 150$, so $150 = 5t + 4.9t^2$ or $4.9t^2 + 5t - 150 = 0$.

$$\begin{aligned} t &= \frac{-5 \pm \sqrt{5^2 - 4(4.9)(-150)}}{2(4.9)} \\ &= \frac{-5 \pm \sqrt{25 + 2940}}{9.8} \\ &= \frac{-5 \pm 54.45}{9.8} \end{aligned}$$

So, $t = 5.05$ or $t = -6.07$ (which is not possible given the physical situation).

Therefore the rock will take 5sec to reach the ground.

Solutions to Selected Exercises 4

1. **i** To verify that the given points lie on the parabola $y = x^2$ substitute in the value of x as follows:

For $A = (1, 1)$ when $x = 1$, $y = (1)^2 = 1$ as required so $A = (1, 1)$ lies on the parabola.

Similarly, for $C' = (0.9, 0.9801)$, when $x = 0.99$, $y = (0.99)^2 = 0.9801$ as required so $C' = (0.99, 0.9801)$ lies on the parabola.

ii The slopes of the chords are calculated using,

$$m = \frac{\text{change in } y}{\text{change in } x}.$$

So, for the chord AB is $m = \frac{1.21-1}{1.1-1} = \frac{0.21}{0.1} = 2.1$.

For the chord AB', $m = \frac{0.81-1}{0.9-1} = \frac{-0.19}{-0.1} = 1.9$.

For the chord AC, $m = \frac{1.0201-1}{1.01-1} = 2.01$.

For the chord AC', $m = \frac{0.9801-1}{0.99-1} = 1.99$.

iii It looks as if the slope of the parabola at A is going to be 2.

2. **i** The answers are given in the notes.

ii **b.** $f(x) = x^2 + x$

Since $a = 1 > 0$, the parabola is upright, and the function has a minimum value. The minimum values occurs when

$$f'(x) = 2x + 1 = 0.$$

That is, when $x = -\frac{1}{2}$ and hence $f(x) = (-\frac{1}{2})^2 - \frac{1}{2} = -\frac{1}{4}$ is the minimum value.

g. $f(x) = 4x - x^2$

Since $a = -1 < 0$, the parabola is upside down and the function has a maximum when $f'(x) = 0$.

$f'(x) = 4 - 2x = 0$ when $x = 2$.

Hence the maximum value of $f(x)$ is $f(x) = 4(2) - (2)^2 = 4$.

3. **c.** $f(x) = 4x^2 - 7x + 6$ so $f'(x) = 8x - 7$. Therefore $f'(0) = 8(0) - 7 = -7$.

4. **ii.** $y = x^2 - 5x + 6$

This parabola is upright and hence has a minimum when

$$\frac{dy}{dx} = 2x - 5 = 0.$$

That is, when $x = \frac{5}{2}$.

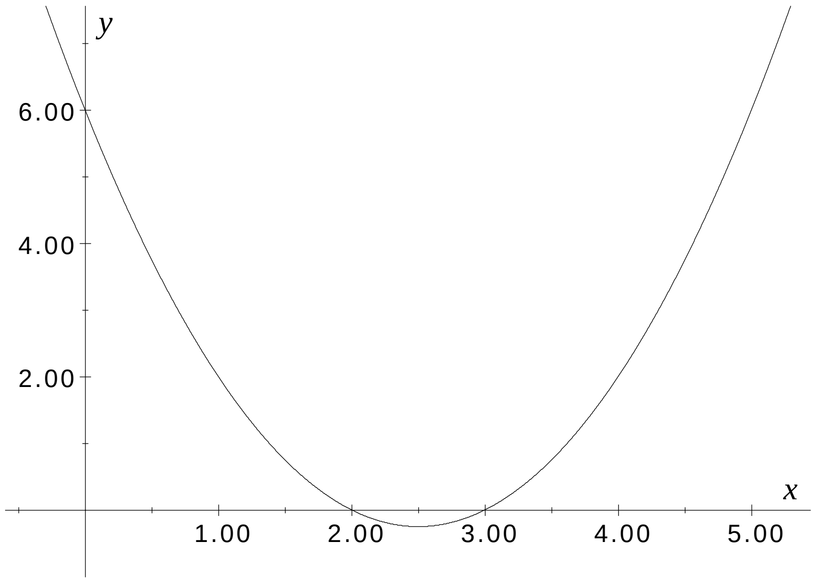
When $x = \frac{5}{2}$, $y = (\frac{5}{2})^2 - 5(\frac{5}{2}) + 6 = -\frac{1}{4}$.

It cuts the y axis when $x = 0$ ie at $y = 6$.

It cuts the x axis when $y = 0$ ie when $x^2 - 5x + 6 = 0$.

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ (x-3)(x-2) &= 0. \end{aligned}$$

That is, when $x = 2$ or $x = 3$.

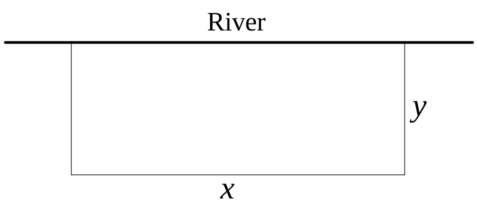


5. iii. To find the gradient of the curve at the given point we need to evaluate the derivative at that point.

$f'(x) = 2x - 3$. When $x = 0$, $f'(x) = -3$ so the gradient of the curve at the point $(0, 1)$ is -3.
The equation of the tangent to the curve at the point $(0, 1)$ is therefore $y = -3x + b$. When $x = 0$, $y = 1$ so $1 = -3(0) + b$ ie $b = 1$.

The equation of the tangent to the curve at $(0, 1)$ is $y = -3x + 1$ or $3x + y - 1 = 0$.

7.



Let x be the length of the side parallel to the river and let y be the length of the other side.

We have 300m of fencing material so $x + 2y = 300$.

The area of the field A is $A = xy$. Substituting $x = 300 - 2y$ we get,

$$A = (300 - 2y)y = 300y - 2y^2.$$

This is a quadratic with $a < 0$ so $A(y)$ has a maximum when $A'(y) = 0$.
Differentiating with respect to y, $A'(y) = 300 - 4y$.

When $A'(y) = 0$, $300 - 4y = 0$, so $y = 75$.

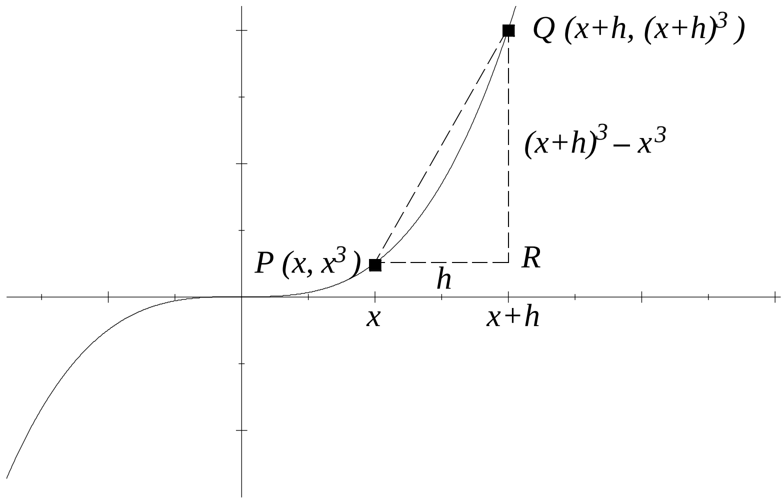
When $y = 75$, $x = 300 - 150 = 150$.

So the area of the field is maximised when the dimensions of the field are 150m by 75m.

Solutions to Selected Exercises 5

1.

i Let P be the point (x, x^3) and Q be the point $(x+h, (x+h)^3)$ on the curve $y = x^3$. Let R be the point $(x+h, x^3)$.



The gradient of the chord PQ is given by

$$\begin{aligned}
\text{gradient of } PQ &= \frac{QR}{PR} \\
&= \frac{(x+h)^3 - x^3}{x+h-x} \\
&= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= 3x^2 + 3xh + h^2.
\end{aligned}$$

ii As $h \to 0$, $3xh + h^2 \to 0$ so the gradient of $PQ \to 3x^2$.

iii

$$\frac{d}{dx}(x^3) = \lim_{h \to 0} \ \text{gradient of } PQ = 3x^2.$$

3. b $f'(x) = 12x^3 + 4x - 1$ so $f'(0) = -1$.

5. Before we differentiate to find any stationary points we will determine where the graph crosses the axes.

When $x = 0$, $y = 0^3(0 - 2) = 0$ so the graph cuts the y-axis at 0.

When $y = 0$, $0 = x^3(x - 2)$ so $x = 0$ or $x = 2$. The graph cuts the x-axis at 0 and 2.

Now $y = x^3(x - 2) = x^4 - 2x^3$, so $y' = 4x^3 - 6x^2$.

Now, $4x^3 - 6x^2 = 2x^2(2x - 3) = 0$ when $x = 0$ or $x = \frac{3}{2}$ so there are two stationary points: $(0, 0)$ and $(\frac{3}{2}, -\frac{27}{16})$.

It is useful to draw up a table as follows to determine the nature of the stationary points.

x	< 0	0	$> 0, \quad < \frac{3}{2}$	$\frac{3}{2}$	$> \frac{3}{2}$
y'	$-ve$	0	$-ve$	0	$+ve$
y	$\searrow$	0	$\searrow$	$-\frac{27}{16}$	$\nearrow$

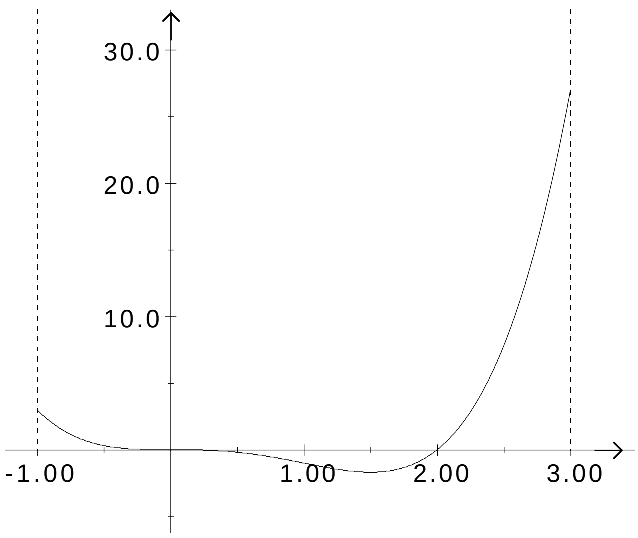
The table tells us that the function is decreasing for $x < 0$ and decreasing in the interval $0 < x < \frac{3}{2}$. Therefore the stationary point at $(0, 0)$ is neither a maximum nor a minimum.

As the function is decreasing in the interval $0 < x < \frac{3}{2}$ and increasing for $x > \frac{3}{2}$, the stationary point at $(\frac{3}{2}, -\frac{27}{16})$ is a minimum.

To complete the picture we need the values of y for $x = -1$ and $x = 3$.

When $x = -1$, $y = (-1)^3(-1 - 2) = (-1)(-3) = 3$. When $x = 3$, $y = (3)^3(3 - 2) = 27$.

Putting all this information together we can now sketch the curve.



7.

i When $x = 1$, $y = (1)^2 + 5(1) - 8 = -2$.

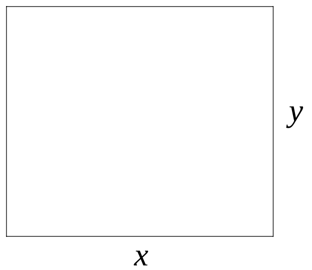
ii $y' = 2x + 5$, so when $x = 1$, $y' = 2(1) + 5 = 7$.
So the value of the derivative at the point $(1, -2)$ is 7.

iii The tangent to the curve at $(1, -2)$ has gradient 7 and passes through $(1, -2)$.

Let $y = 7x + b$. When $x = 1$, $y = -2$ so $-2 = 7(1) + b$ ie $b = -9$.

The equation of the tangent to the curve $y = x^2 + 5x - 7$ at the point $(1, -2)$ is $y = 7x - 9$ or $7x - y - 9 = 0$.

9. Let the perimeter of the rectangle be $2p$ where p is a constant. Let the sides of the rectangle be of length x and y.



Then $2x + 2y = 2p$ ie $y = p - x$. (Making the perimeter $2p$ instead of p makes the algebra a bit easier.)

The area of the rectangle A is given by $A = xy = x(p - x) = px - x^2$.

To maximise the area we need to differentiate A with respect to x and set the derivative equal to 0.
That is, $A' = p - 2x = 0$ (remember that p is a constant).

When $A' = 0$, $p - 2x = 0$ ie $x = \frac{p}{2}$. We know we have a maximum when $x = \frac{p}{2}$ as the coefficient of the x^2 term in $A = px - x^2$ is negative.

When $x = \frac{p}{2}$, $y = p - \frac{p}{2} = \frac{p}{2}$ so the rectangle with the maximum area for a given perimeter is a square.

Solutions to Selected Exercises 6

2. iii $y = x^3 - 12x + 12$

$y' = 3x^2 - 12$. There are stationary points when $y' = 0$,

$$y' = 3x^2 - 12 = 3(x^2 - 4) = 3(x-2)(x+2) = 0 \quad \text{ie when} \quad x = \pm 2.$$

So, $(-2, 28)$ and $(2, -4)$ are stationary points.

We will investigate the nature of the stationary points by drawing up a table.

x	< -2	-2	$> -2, \quad < 2$	2	> 2
y'	$+ve$	0	$-ve$	0	$+ve$
y	$\nearrow$	28	$\searrow$	-4	$\nearrow$

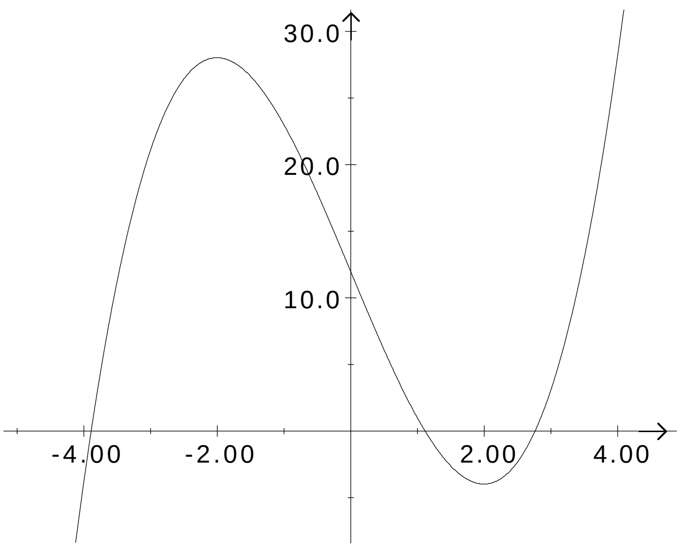
The table tells us that the function is increasing for $x < -2$ and decreasing for $-2 < x < 2$ so the point $(-2, 28)$ is a maximum. The function is decreasing for $-2 < x < 2$ and increasing for $x > 2$, so the point $(2, -4)$ is a minimum.

A point of inflection occurs when $\frac{d^2y}{dx^2} = 0$ and there is a change of concavity.

$\frac{d^2y}{dx^2} = 6x = 0$ ie $x = 0$.

x	< 0	0	> 0
y''	$-ve$	0	$+ve$
y	concave down	$+12$	concave up

From the table we see that when $x = 0$, $\frac{d^2y}{dx^2} = 0$ and there is a change of concavity from concave down to concave up. Therefore, there is a point of inflection at $(0, 12)$.



vi $y = (x-1)^3$

We must expand out the bracket in order to differentiate.

$$y = (x-1)^3 = (x-1)(x^2-2x+1) = x^3 - 2x^2 + x - x^2 + 2x - 1 = x^3 - 3x^2 + 3x - 1.$$

Differentiating and setting the derivative equal to 0, we get

$$\frac{dy}{dx} = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2 = 0 \quad \text{ie} \quad x = 1.$$

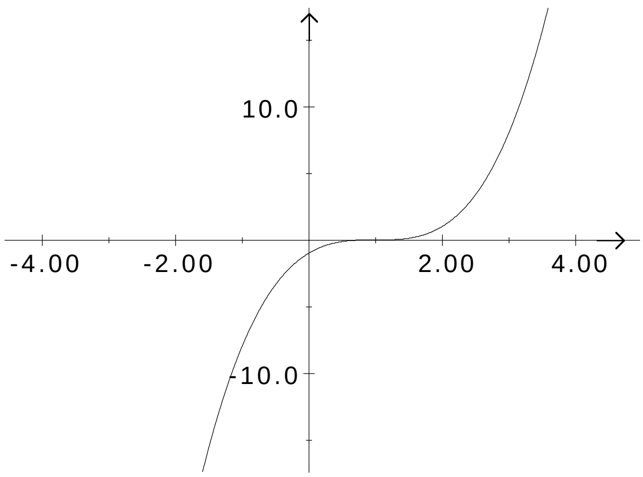
x	< 1	1	> 1
y'	$+ve$	0	$+ve$
y	$\nearrow$	0	$\nearrow$

Since the function is increasing for $x < 1$ and increasing for $x > 1$, the point $(1, 0)$ is a (horizontal) point of inflection.

Differentiating again to find the second derivative we get,

$$\frac{d^2y}{dx^2} = 6x - 6 = 6(x-1).$$

This is equal to 0 when $x = 1$ as before, so there are no other points of inflection.



vii $y = x^4 - 2x^2$

$y' = 4x^3 - 4x = 4x(x^2-1) = 0$ when $x = 0$ or $x \pm 1$.

x	< -1	-1	$> -1, < 0$	0	$> 0, < 1$	1	> 1
y'	$-ve$	0	$+ve$	0	$-ve$	0	$+ve$
y	$\searrow$	-1	$\nearrow$	0	$\searrow$	-1	$\nearrow$

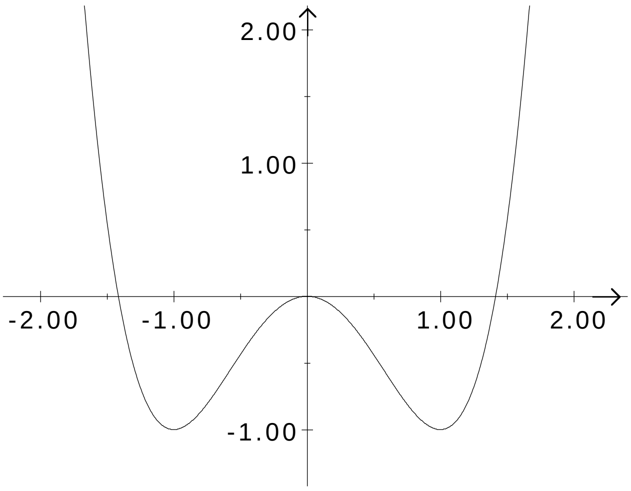
As the function is decreasing for $x < -1$ and increasing for $-1 < x < 0$, the point $(-1, -1)$ is a minimum. As the function is increasing for $-1 < x < 0$ and decreasing for $0 < x < 1$, the point $(0, 0)$ is a maximum. As the function is decreasing for $0 < x < 1$ and increasing for $x > 1$, the point $(1, -1)$ is a minimum.

Points of inflection occur when $y'' = 0$ and the concavity changes.

$y'' = 12x^2 - 4 = 4(3x^2 - 1) = 0$ ie when $x = \pm\frac{1}{\sqrt{3}}$.

x	$< -\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$> -\frac{1}{\sqrt{3}}, \quad < \frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$> \frac{1}{\sqrt{3}}$
y''	$+ve$	0	$-ve$	0	$+ve$
y	concave up	$-\frac{5}{9}$	concave down	$-\frac{5}{9}$	concave up

We see from the table that $(-\frac{1}{\sqrt{3}}, -\frac{5}{9})$ and $(\frac{1}{\sqrt{3}}, -\frac{5}{9})$ are points of inflection.



3.

i The velocity at time t is given by

$$\frac{ds}{dt} = 24 - 9.8t.$$

When $t = 0$, $\frac{ds}{dt} = 24$, so the initial velocity of the rock is 24 m/sec.

ii The rock achieves its maximum height when the velocity is zero. $\frac{ds}{dt} = 0$ when $t = \frac{24}{9.8} = 2.45$ sec.

The maximum height of the rock is $s = 24(2.45) - 4.9(2.45)^2 = 29.4$ metres.

iii When the rock hits the ground $s = 0$, so $24t - 4.9t^2 = t(24 - 4.9t) = 0$ ie $t = 0$ or $t = \frac{24}{4.9} = 4.90$.

The rock takes about 5 seconds to fall back to the ground.

5. The velocity is zero when $\frac{ds}{dt} = 3t^2 - 8t - 3 = (3t+1)(t-3) = 0$ ie when $t = 3$. (We can discard $t = -\frac{1}{3}$ as it is negative.)

The acceleration of the body is given by

$$\frac{d^2s}{dx^2} = 6t - 8.$$

When $t = 3$, $\frac{d^2s}{dx^2} = 6(3) - 8 = 10$.

So the acceleration of the body is 10 m/sec^2 when the velocity of the body is zero.

Solutions to Selected Exercises 7

1. iii

$$\begin{aligned} uv^{-2} \times \frac{u^{-1}}{v} &= uv^{-2} \times u^{-1}v^{-1} \\ &= uu^{-1}v^{-2}v^{-1} \\ &= v^{-3}. \end{aligned}$$

2. iii

$$ab^{-1} + bc^{-1} = \frac{a}{b} + \frac{b}{c} = \frac{2}{3} + \frac{3}{\frac{1}{2}} = \frac{2}{3} + 6 = 6\frac{2}{3}.$$

3. iv

$$\left(\frac{4}{9}\right)^{-\frac{3}{2}} = \frac{1}{(\frac{4}{9})^{\frac{3}{2}}} = \left(\frac{9}{4}\right)^{\frac{3}{2}} = \left(\left(\frac{9}{4}\right)^{\frac{1}{2}}\right)^{3} = \left(\frac{3}{2}\right)^{3} = \frac{27}{8}.$$

4. ii

$$m^{\frac{1}{3}}(m^{\frac{1}{3}} + m^{-\frac{1}{3}}) = m^{\frac{1}{3}}m^{\frac{1}{3}} + m^{\frac{1}{3}}m^{-\frac{1}{3}} = m^{\frac{2}{3}} + m^{0} = m^{\frac{2}{3}} + 1.$$

5. ii

$$\frac{t^2t^{-3}}{\sqrt[3]{t}} = \frac{t^2t^{-3}}{t^{\frac{1}{3}}} = t^2t^{-3}t^{-\frac{1}{3}} = t^{2-3-\frac{1}{3}} = t^{-\frac{4}{3}}.$$

iv

$$\frac{m^{\frac{3}{2}}\sqrt[3]{m}}{m^{-3}} = \frac{m^{\frac{3}{2}}m^{\frac{1}{3}}}{\frac{1}{m^3}} = m^{\frac{3}{2}}m^{\frac{1}{3}}m^3 = m^{\frac{3}{2}+\frac{1}{3}+3} = m^{\frac{29}{6}}.$$

viii

$$\frac{1}{x^{-2}\sqrt{x^{-1}}} = \frac{1}{x^{-2}(x^{-1})^{\frac{1}{2}}} = \frac{1}{x^{-2}x^{-\frac{1}{2}}} = \frac{1}{x^{-\frac{5}{2}}} = x^{\frac{5}{2}}.$$

6. vi

$$f(x) = 1 + 2\sqrt{x} = 1 + 2x^{\frac{1}{2}} \quad \text{so} \quad f'(x) = 2\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \frac{1}{x^{\frac{1}{2}}} = \frac{1}{\sqrt{x}}.$$

7. iii

$$y = x^2 + \frac{2}{x^2} = x^2 + 2x^{-2} \quad \text{so} \quad \frac{dy}{dx} = 2x + 2(-2)x^{-3} = 2x - \frac{4}{x^3}.$$

8. iv

$$h(u) = 13u^2 - 5u\sqrt{u} = 13u^2 - 5uu^{\frac{1}{2}} = 13u^2 - 5u^{\frac{3}{2}}$$
$$\text{so} \quad h'(u) = 26u - \frac{15}{2}u^{\frac{1}{2}} = 26u - \frac{15}{2}\sqrt{u}.$$

vi

$$f(x) = \frac{5}{x^3} + \frac{1}{x} = 5x^{-3} + x^{-1} \quad \text{so} \quad f'(x) = -15x^{-4} - x^{-2} = -\frac{15}{x^4} - \frac{1}{x^2}.$$

9. iv

$$f(x) = x^{\frac{1}{2}} + x^{-\frac{1}{2}} \quad \text{so} \quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} \quad \text{and} \quad f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{5}{2}}.$$

viii

$$y = \frac{5x^3 + 2x^{\frac{1}{2}}}{x} = 5x^2 + 2x^{-\frac{1}{2}} \quad \text{so} \quad \frac{dy}{dx} = 10x - x^{-\frac{3}{2}} \quad \text{and} \quad \frac{d^2y}{dx^2} = 10 + \frac{3}{2}x^{-\frac{5}{2}}.$$

10. i Let h be the height of the tank. The volume (V) of the tank is $V = x^2h = 32$.

$$\text{So,} \quad h = \frac{32}{x^2}.$$

ii Let the area of the sheet metal be A. Then

$$A = 4xh + x^2 = 4x\frac{32}{x^2} + x^2 = 128x^{-1} + x^2.$$

To maximise A, differentiate with respect to x and set the derivative equal to 0.

$$\frac{dA}{dx} = -128x^{-2} + 2x = -\frac{128}{x^2} + 2x = \frac{-128 + 2x^3}{x^2}$$
$$\text{so} \quad \frac{dA}{dx} = 0 \quad \text{when} \quad 2x^3 = 128 \quad \text{ie} \quad x = 4.$$

When $x < 4$, $\frac{dA}{dx} < 0$ (use $x = 1$ as a test point) and when $x > 4$ $\frac{dA}{dx} > 0$ (use $x = 5$ as a test point) so we have a minimum when $x = 4$.

When $x = 4$, $A = \frac{128}{4} + 4^2 = 32 + 16 = 48$.

So the least area of sheet metal is 48 m^2.

Solutions to Selected Exercises 8

4. ii

$y = \dfrac{e^x + x}{2} = \dfrac{1}{2}(e^x + x)$ so $\dfrac{dy}{dx} = \dfrac{1}{2}(e^x + 1)$.

The derivative of e^x is e^x and the derivative of x is 1.

Also, note that we have used the rule that the derivative of a constant $\times$ a function is equal to the constant $\times$ the derivative of the function. (See page 15 of the notes.)

5. $P = Ae^{0.02t}$

When $t = 0$ (in 1970) $P = 100,000$ so substituting we get $100000 = Ae^{(0.02)(0)} = Ae^0 = A$.

So, $P = 100000e^{0.02t}$.

In 1980 $t = 10$, so $P = 100000e^{(0.02)(10)} = 100000(1.2214) = 122140$.

Therefore, the population in 1980 will be 122,000 to the nearest thousand.

6. ii $A = A_0e^{-0.00012t}$

When $t = 6000$, $A = A_0e^{(-0.00012)(6000)} = A_0e^{-0.72} = 0.49A_0$.

There is about 0.5 of the initial amount remaining after 6000 years.

7. iii $(x^5 + 1)^3$

Let $u = f(x) = x^5 + 1$ and $g(u) = u^3$, then $g(f(x)) = g(x^5 + 1) = (x^5 + 1)^3$.

iv $\dfrac{1}{3 - e^x}$

Let $u = f(x) = 3 - e^x$ and $g(u) = \dfrac{1}{u}$, then $g(f(x)) = g(3 - e^x) = \dfrac{1}{3 - e^x}$.

Solutions to Selected Exercises 9

1. **ii** $y = \sqrt{5 - x^2}$

First rewrite the equation as $y = (5 - x^2)^{\frac{1}{2}}$.

This is a composite function with $u = f(x) = 5 - x^2$ and $g(u) = u^{\frac{1}{2}}$,

so $g(f(x)) = (5 - x^2)^{\frac{1}{2}}$.

So,

$$\frac{dy}{dx} = \underbrace{\frac{1}{2}(5 - x^2)^{-\frac{1}{2}}}_{g'(f(x))} \times \underbrace{(-2x)}_{f'(x)}.$$

2. **ii** $y = \dfrac{4x}{x^2 + 1}$

This is quotient so we will use the quotient rule.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(x^2+1)\frac{d}{dx}(4x) - (4x)\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\
&= \frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2} \\
&= \frac{4x^2 + 4 - 8x^2}{(x^2+1)^2} \\
&= \frac{4 - 4x^2}{(x^2+1)^2}.
\end{aligned}$$

3. **b** $y = 2x^2(x^2 - 5)$

Using the product rule we get

$$\frac{dy}{dx} = 2x^2 \times \frac{d}{dx}(x^2 - 5) + (x^2 - 5) \times \frac{d}{dx}(2x^2) = 2x^2 \times 2x + (x^2 - 5) \times 4x = 4x^3 + 4x^3 - 20x = 8x^3 - 20x.$$

h $y = \dfrac{1}{(2x+3)^5}$

First we will rewrite the equation as $y = (2x + 3)^{-5}$ and use the chain rule. So,

$$\frac{dy}{dx} = -5(2x+3)^{-6}\frac{d}{dx}(2x+3) = -5(2x+3)^{-6}(2) = -10(2x+3)^{-6} = \frac{-10}{(2x+3)^6}.$$

j $y = (x^2+1)(x^2-6)$

Here we will use the product rule.

$$\begin{aligned}\frac{dy}{dx} &= (x^2+1)\frac{d}{dx}(x^2-6) + (x^2-6)\frac{d}{dx}(x^2+1)\\ &= (x^2+1)(2x) + (x^2-6)(2x)\\ &= 2x^3+2x+2x^3-12x = 4x^3-10x.\end{aligned}$$

s $y = \dfrac{x}{\sqrt{x^2+1}}$

First we rewrite the equation as $y = \dfrac{x}{(x^2+1)^{\frac{1}{2}}}$ and use the quotient rule to differentiate. We'll need the chain rule too.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2+1)^{\frac{1}{2}} \times \frac{d}{dx}(x) - x \times \frac{d}{dx}\left((x^2+1)^{\frac{1}{2}}\right)}{\left((x^2+1)^{\frac{1}{2}}\right)^2}\\ &= \frac{(x^2+1)^{\frac{1}{2}} \times 1 - x \times \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x}{x^2+1}\\ &= \frac{(x^2+1)^{\frac{1}{2}} - \frac{x^2}{(x^2+1)^{\frac{1}{2}}}}{x^2+1}\\ &= \frac{\frac{(x^2+1)-x^2}{(x^2+1)^{\frac{1}{2}}}}{x^2+1}\\ &= \frac{1}{(x^2+1)^{\frac{3}{2}}}.\end{aligned}$$

4. **ii** $f(x) = e^{-2x}$

$$f'(x) = e^{-2x} \times \frac{d}{dx}(-2x) = -2e^{-2x}.$$

vi $f(x) = e^{x^2-2x+7}$

$$f'(x) = e^{x^2-2x+7} \times \frac{d}{dx}(x^2-2x+7) = (2x-2)e^{x^2-2x+7} = 2(x-1)e^{x^2-2x+7}.$$

viii $f(x) = x^2e^x$

$$f'(x) = x^2 \times \frac{d}{dx}(e^x) + e^x \times \frac{d}{dx}(x^2) = x^2e^x + 2xe^x = xe^x(x+2) = x(x+2)e^x.$$

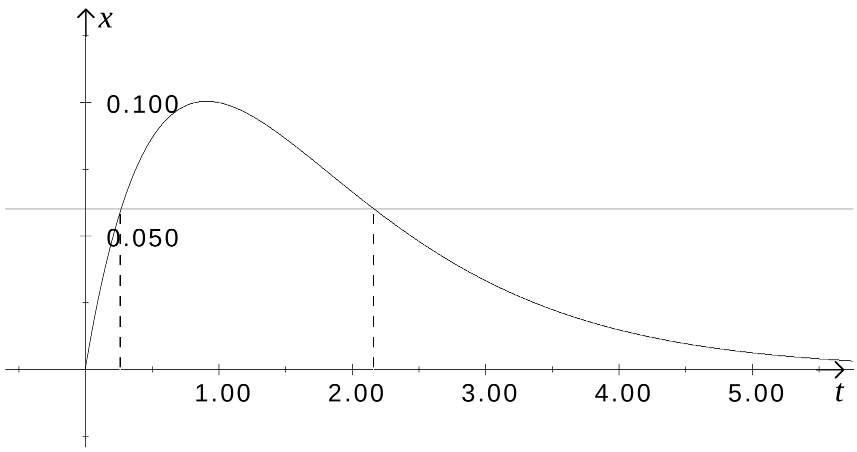
5. The maximum concentration of the drug in the blood occurs when the derivative of $x = 0.3te^{-1.1t}$ equals zero.

$$x' = 0.3e^{-1.1t} + 0.3t(-1.1)e^{-1.1t} = 0.3e^{-1.1t}(1 - 1.1t) = 0 \quad \text{ie when} \quad t = \frac{1}{1.1} = \dot{9}\dot{0}.$$

Note that $e^{-1.1t} > 0$ for all values of t. The following table confirms we have a maximum when $t = \dot{9}\dot{0}$.

t	$< \dot{9}\dot{0}$	$\dot{9}\dot{0}$	$> \dot{9}\dot{0}$
x'	$+ve$	0	$-ve$
y	$\nearrow$	$\frac{0.3e^{-1}}{1.1}$	$\searrow$

The maximum value of $\frac{0.3e^{-1}}{1.1}$ units is achieved when $t = \dot{9}\dot{0}$ ie when $t = 54.5$ minutes.



From my sketch, the drug will kill germs between about 0.25 hours and 2.15 hours after it is taken. (This is when the graph of $x = 0.3te^{-1.1t}$ is above the line $x = 0.06$.) So, the length of time the drug is able to kill germs is about 1.9 hours.

6. $y = e^{-x^2}$

The first thing to note about this function is that it is always positive. ($e^{-x^2} > 0$ for all values of x.) So, the graph of $y = e^{-x^2}$ is above the x-axis and never crosses it.

$$\frac{dy}{dx} = e^{-x^2} \times \frac{d}{dx}(-x^2) = -2xe^{-x^2}.$$

This is equal to zero when $x = 0$. So, $(0, 1)$ is the only stationary point.

x	< 0	0	> 0
y'	$+ve$	0	$-ve$
y	$\nearrow$	1	$\searrow$

The table tells us that $(0,1)$ is a maximum.

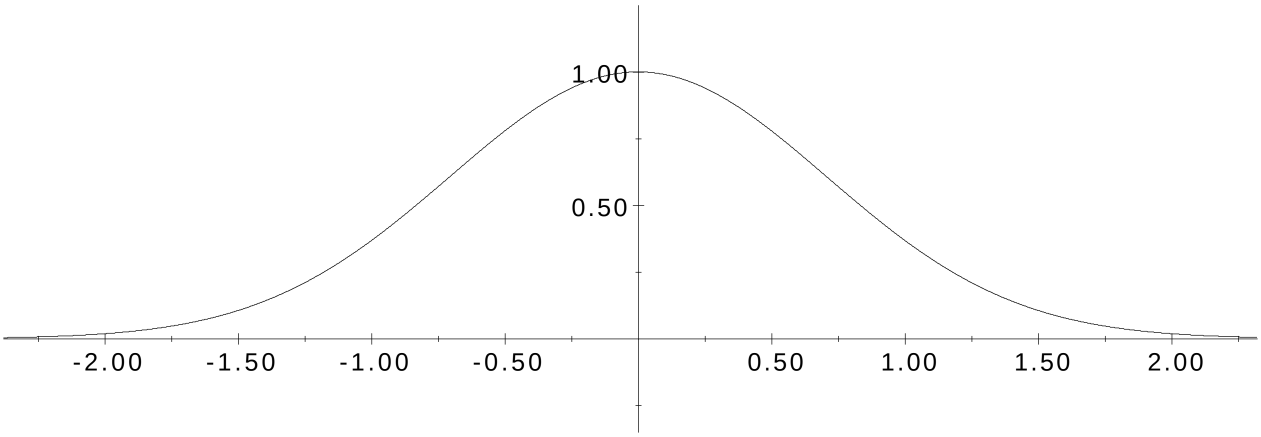
We differentiate again to see if there are any points of inflection.

$$\frac{d^2y}{dx^2} = (-2x)e^{-x^2}(-2x) + e^{-x^2}(-2) = 2e^{-x^2}(2x^2 - 1).$$

This equals zero when $x^2 = \frac{1}{2}$, ie when $x = \pm\frac{1}{\sqrt{2}}$.

x	$< -\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$> -\frac{1}{\sqrt{2}}, < \frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$> \frac{1}{\sqrt{2}}$
y''	$+ve$	0	$-ve$	0	$+ve$
y	concave up	$e^{-\frac{1}{2}}$	concave down	$e^{-\frac{1}{2}}$	concave up

The table tells us that there are two points of inflection at $(-\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}})$ and $(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}})$.



Note that the graph does not touch the x-axis but gets closer and closer to it as x gets large in magnitude.

Solutions to Selected Exercises 10

3. ii $y = \sin(x)\cos(x)$

$$\frac{dy}{dx} = \sin x \times \frac{d}{dx}(\cos x) + \cos x \times \frac{d}{dx}(\sin x) = \sin x(-\sin x) + \cos x(\cos x) = \cos^2 x - \sin^2 x.$$

When we differentiate again we can either use the chain rule or treat $\cos^2 x$ and $\sin^2 x$ as products, ie $\cos^2 x = \cos x \cos x$.

Using the product rule we get,

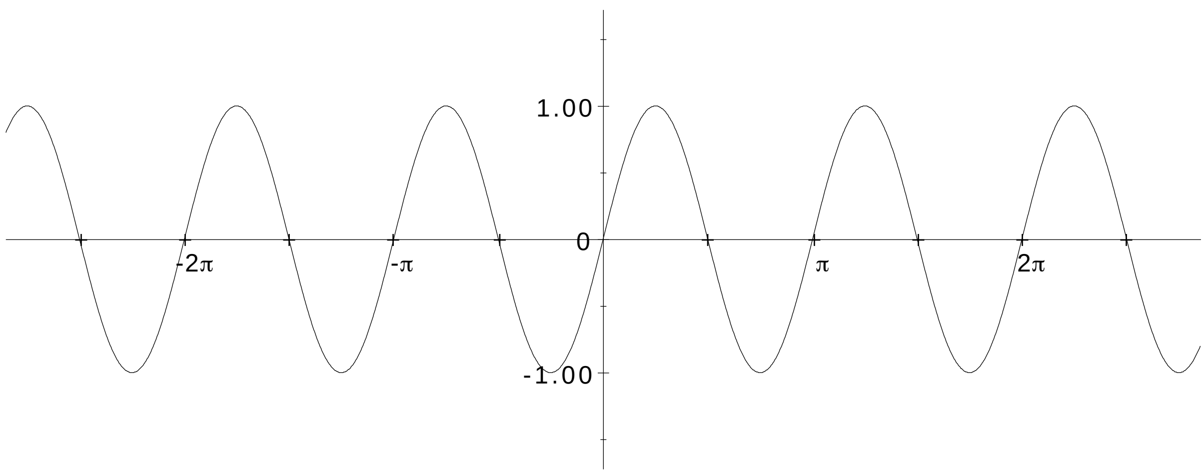
$$\begin{aligned}
\frac{d^2y}{dx^2} &= \cos x \times \frac{d}{dx}(\cos x) + \cos x \times \frac{d}{dx}(\cos x) - \left(\sin x \times \frac{d}{dx}(\sin x) + \sin x \times \frac{d}{dx}(\sin x)\right) \\
&= \cos x(-\sin x) + \cos x(-\sin x) - \sin x(\cos x) - \sin x(\cos x) \\
&= -4\sin x \cos x.
\end{aligned}$$

v $y = x\cos x$

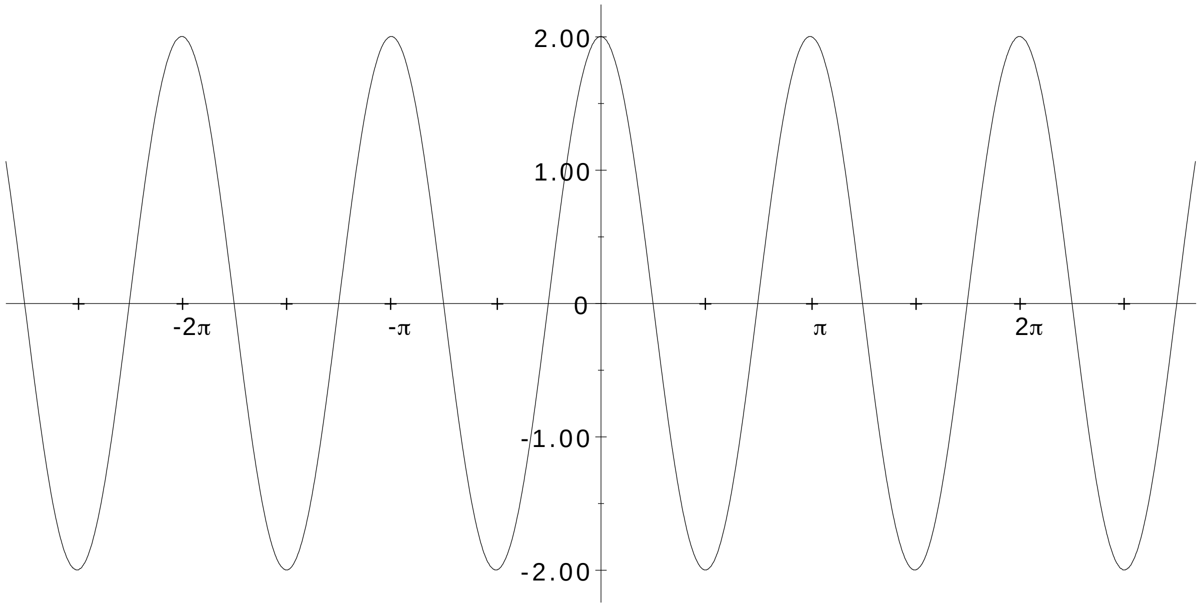
$$\frac{dy}{dx} = x \times \frac{d}{dx}(\cos x) + \cos x \times \frac{d}{dx}(x) = -x\sin x + \cos x.$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= (-x) \times \frac{d}{dx}(\sin x) + \sin x \times \frac{d}{dx}(-x) - \sin x \\
&= (-x)(\cos x) + \sin x(-1) - \sin x \\
&= -x\cos x - 2\sin x.
\end{aligned}$$

4. A sketch of the function $y = \sin 2x$ is given below. Notice that its amplitude is still 1 but its period is π.



This is a sketch of the derivative of $y = \sin 2x$. It is a cos function with amplitude 2, and period π. How close did you get?



5. ii $y = \cos(x + x^2)$

Here $u = f(x) = x + x^2$ and $g(u) = \cos u$, so

$$\frac{dy}{dx} = \underbrace{-\sin(x + x^2)}_{g'(f(x))} \times \frac{d}{dx}(x + x^2) = -(1 + 2x)\sin(x + x^2).$$

iii $y = \sin(x^2)$

Here $u = f(x) = x^2$ and $g(u) = \sin u$, so

$$\frac{dy}{dx} = \underbrace{\cos(x^2)}_{g'(f(x))} \times \frac{d}{dx}(x^2) = 2x\cos(x^2).$$

vi $y = 2\sin(x + \pi)$

Here $u = f(x) = x + \pi$ and $g(u) = 2\sin u$, so

$$\frac{dy}{dx} = \underbrace{2\cos(x + \pi)}_{g'(f(x))} \times \frac{d}{dx}(x + \pi) = 2\cos(x + \pi).$$

Remember that π is a constant so its derivative is zero.

6. iii $y = \sin(3x)\cos(x^2)$

$$\begin{aligned}\frac{dy}{dx} &= \sin(3x) \times \frac{d}{dx}(\cos(x^2)) + \cos(x^2) \times \frac{d}{dx}(\sin(3x)) \\ &= \sin(3x)(-\sin(x^2) \times (2x)) + \cos(x^2)(3\cos(3x)) \\ &= -2x\sin(3x)\sin(x^2) + 3\cos(3x)\cos(x^2).\end{aligned}$$

v $y = \dfrac{3x}{1+\cos x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+\cos x) \times \frac{d}{dx}(3x) - 3x \times \frac{d}{dx}(1+\cos x)}{(1+\cos x)^2} \\ &= \frac{3(1+\cos x) - 3x(-\sin x)}{(1+\cos x)^2} \\ &= \frac{3(1+\cos x + x\sin x)}{(1+\cos x)^2}.\end{aligned}$$

vi $y = \dfrac{\sin x}{\cos x} = \tan x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\cos x \times \frac{d}{dx}(\sin x) - \sin x \times \frac{d}{dx}(\cos x)}{(\cos x)^2} \\ &= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} && \sin^2 x + \cos^2 x = 1 \\ &= \frac{1}{\cos^2 x} && \frac{1}{\cos x} = \sec x \\ &= \sec^2 x.\end{aligned}$$

7. b $f(x) = \cos(x^2 - 2x + 1)$

$$f'(x) = -\sin(x^2 - 2x + 1) \times \frac{d}{dx}(x^2 - 2x + 1) = -(2x-2)\sin(x^2 - 2x + 1).$$

d $f(x) = \sin(x^{-1})$

$$f'(x) = \cos(x^{-1}) \times \frac{d}{dx}(x^{-1}) = \cos(x^{-1}) \times (-x^{-2}) = -x^{-2}\cos(x^{-1}).$$

e $f(x) = x^2\cos(x^2 + 4)$

$$\begin{aligned}f'(x) &= x^2 \times \frac{d}{dx}(\cos(x^2+4)) + \cos(x^2+4) \times \frac{d}{dx}(x^2) \\ &= x^2(-\sin(x^2+4) \times (2x)) + \cos(x^2+4) \times (2x) \\ &= -2x^3\sin(x^2+4) + 2x\cos(x^2+4).\end{aligned}$$

h $f(x) = \sin(\cos x)$

$$f'(x) = \cos(\cos x) \times \frac{d}{dx}(\cos x) = \cos(\cos x) \times (-\sin x) = -\cos(\cos x) \cdot \sin x.$$

8. ii Write $\cos(x - \pi) = \cos(x + (-\pi))$ and use the identity on page 37 of the notes.

$$\cos(x - \pi) = \cos(x + (-\pi)) = \cos x \cos(-\pi) - \sin x \sin(-\pi) = -\cos x.$$

Note that $\cos(-\pi) = \cos \pi = -1$ and $\sin(-\pi) = -\sin \pi = 0$.

iv Write $\cos(2\pi - x) = \cos(2\pi + (-x))$.

$$\cos(2\pi - x) = \cos 2\pi \cos(-x) - \sin 2\pi \sin(-x) = \cos(-x) = \cos x.$$

Note that $\cos 2\pi = 1$, $\sin 2\pi = 0$ and $\cos(-x) = \cos x$.

Solution to Selected Exercises 11

1. i

$$e^{3\ln x} = e^{\ln x^3} = x^3.$$

We used the log rule $\ln a^b = b\ln a$ and the fact that e^x and $\ln x$ are inverses.

iv

$$\ln x^3 - \ln x = \ln\left(\frac{x^3}{x}\right) = \ln x^2.$$

2. ii

$$\begin{aligned}
\ln(3+x) &= 1\\
e^{\ln(3+x)} &= e^1\\
3+x &= e\\
x &= e-3.
\end{aligned}$$

iv

$$\begin{aligned}
\ln(5x-6) &= 2\\
e^{\ln(5x-6)} &= e^2\\
5x-6 &= e^2\\
5x &= e^2+6\\
x &= \frac{e^2+6}{5}.
\end{aligned}$$

3. iv

$$\begin{aligned}
e^{x^2} &= 10\\
\ln(e^{x^2}) &= \ln 10\\
x^2 &= \ln 10\\
x &= \pm\sqrt{\ln 10}.
\end{aligned}$$

4. iii

$$\frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x) = \ln(x+1)^{\frac{1}{2}} - \ln(x)^{\frac{1}{2}} = \ln\left(\frac{(x+1)^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right) = \ln\left(\sqrt{\frac{x+1}{x}}\right).$$

5. i c

$$\begin{aligned}
\ln\left(\frac{x^2\sqrt{x+1}}{\sqrt[3]{3x+4}}\right) &= \ln(x^2) + \ln(\sqrt{x+1}) - \ln(\sqrt[3]{3x+4})\\
&= 2\ln x + \ln(x+1)^{\frac{1}{2}} - \ln(3x+4)^{\frac{1}{3}}\\
&= 2\ln x + \frac{1}{2}\ln(x+1) - \frac{1}{3}\ln(3x+4).
\end{aligned}$$

ii c

$$\begin{aligned}\frac{d}{dx}\left(\ln\left(\frac{x^2\sqrt{x+1}}{\sqrt[3]{3x+4}}\right)\right) &= \frac{d}{dx}(2\ln x) + \frac{d}{dx}\left(\frac{1}{2}\ln(x+1)\right) - \frac{d}{dx}\left(\frac{1}{3}\ln(3x+4)\right)\\ &= \left(2\times\frac{1}{x}\right) + \left(\frac{1}{2}\times\frac{1}{x+1}\right) - \left(\frac{1}{3}\times\frac{1}{3x+4}\times 3\right)\\ &= \frac{2}{x} + \frac{1}{2(x+1)} - \frac{1}{3x+4}.\end{aligned}$$

6. iii

$$\frac{d}{dx}(\ln(x^2)) = 2\frac{d}{dx}(\ln x) = 2\times\frac{1}{x} = \frac{2}{x}.$$

v

$$\frac{d}{dx}(\cos x\times\ln x) = \cos x\times\frac{1}{x} + \ln x\times(-\sin x) = \frac{\cos x}{x} - \sin x\times\ln x.$$

vi

$$\frac{d}{dx}(\sqrt{\ln x}) = \frac{d}{dx}((\ln x)^{\frac{1}{2}}) = \frac{1}{2}(\ln x)^{-\frac{1}{2}}\times\frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}.$$

7. Let P be the population of the city after t years. Then $P = Ae^{kt}$.
When $t = 0$ (in 1970) $P = 2\times 10^6$ so, $2\times 10^6 = Ae^0 = A$ ie $P = 2\times 10^6 e^{kt}$.

When $t = 10$ $P = 2.5\times 10^6$,

$$\begin{aligned}2.5\times 10^6 &= 2\times 10^6 e^{10k}\\ \frac{2.5}{2} &= e^{10k}\\ 10k &= \ln 1.25\\ k &= \frac{\ln 1.25}{10}\\ k &= 0.0223.\end{aligned}$$

Therefore, $P = 2\times 10^6 e^{0.022t}$.

Solutions to Selected Exercises 12

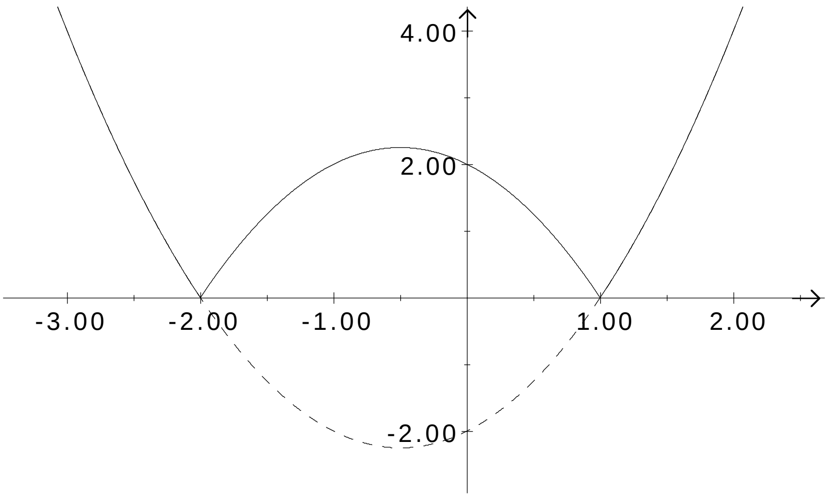
1.

$$||2| - |-5|| = |2-5| = |-3| = 3.$$

2. **iii** If $|2+x| = 5$ then $2+x=5$ or $2+x=-5$. So, $x=3$ or $x=-7$.

3. **ii** $y = |x^2+x-2|$

The easiest way to draw this graph is to sketch the graph of the function $y = x^2+x-2$ and reflect the part below the x-axis in the x-axis.



4. **i**

The gradient of the line is $m = \dfrac{4-2}{-1-1} = -1.$

Let $y = -x+b$. When $x = 1$ $y = 2$ so, $2 = -1(1)+b$ ie $b = 3$.

The equation of the line is $y = -x+3$ or $x+y-3 = 0$.

5. **ii** To find the point(s) of intersection of the two curves we write the equation of the line as $y = x-1$, let $\dfrac{1}{x-1} = x-1$ and solve for x.

$$\begin{aligned}\frac{1}{x-1} &= x-1\\ (x-1)^2 &= 1\\ x^2-2x+1 &= 1\\ x^2-2x &= 0\\ x(x-2) &= 0\end{aligned}$$

Therefore, $x = 0$ or $x = 2$. Substitute to find y.

The two curves intersect at $(0,-1)$ and $(2,1)$.

7. Let the length of the sides of the rectangle be x and y. Then the perimeter of the rectangle is $2x + 2y = 40$ and the area, A, is $A = xy = 96$.

Substituting $y = 20 - x$ in A we get,

$$A = x(20 - x) = 20x - x^2 = 96 \quad \text{or} \quad x^2 - 20x + 96 = 0.$$

That is, $x^2 - 20x + 96 = (x - 8)(x - 12) = 0$, ie $x = 8$ or $x = 12$.

Therefore the dimensions of the rectangle are 8 cms by 12 cms.

8. **a**

$$v^3\sqrt{\frac{1}{v}} = v^3\left(\frac{1}{v}\right)^{\frac{1}{2}} = v^3(v^{-1})^{\frac{1}{2}} = v^3(v^{-\frac{1}{2}}) = v^{\frac{5}{2}}.$$

d

$$\sqrt[3]{\frac{x^7}{x^3}} = \left(\frac{x^7}{x^3}\right)^{\frac{1}{3}} = \frac{x^{\frac{7}{3}}}{x^{\frac{3}{3}}} = x^{\frac{7}{3}}x^{-1} = x^{\frac{4}{3}}.$$

h

$$\frac{v^4}{v^{\frac{3}{2}}v^{-2}} = v^4v^{-\frac{3}{2}}v^{-(-2)} = v^{4-\frac{3}{2}+2} = v^{\frac{9}{2}}.$$

l

$$\frac{\sqrt[4]{x}}{x^2} = \frac{x^{\frac{1}{4}}}{x^2} = x^{\frac{1}{4}}x^{-2} = x^{-\frac{7}{4}}.$$

10. **iv**

$$\frac{d}{dx}(x\sin x) = x(\cos x) + \sin x(1) = x\cos x + \sin x.$$

viii Let $u = f(x) = \sin x$ and $g(u) = e^u$ so,

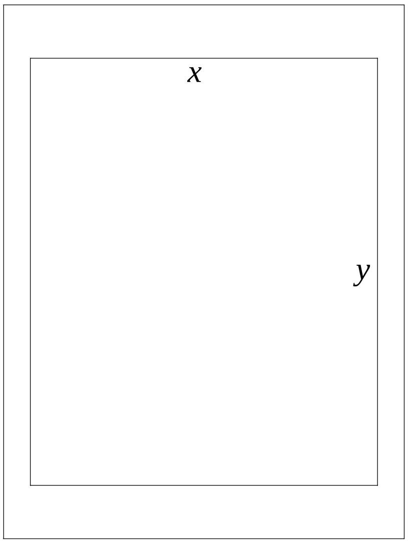
$$\frac{d}{dx}(e^{\sin x}) = \underbrace{e^{\sin x}}_{g'(f(x))} \times \frac{d}{dx}(\sin x) = e^{\sin x}\cos x.$$

ix Let $u = f(x) = \frac{1}{x}$ and $g(u) = \tan u$ so,

$$\frac{d}{dx}(\tan\frac{1}{x}) = \underbrace{\sec^2\frac{1}{x}}_{g'(f(x))} \times \frac{d}{dx}(x^{-1}) = -\frac{\sec^2\frac{1}{x}}{x^2}.$$

Note that $\dfrac{d}{du}(\tan u) = \sec^2 u$.

11. Let the internal dimensions of the advertisement be x and y as shown in the diagram.



Then the overall dimensions are $x+4$ and $y+8$. We also know that the internal area is 50 cm^2, ie $xy = 50$.

Let A be the overall area of the advertisement so

$$A = (x+4)(y+8) = (x+4)\left(\frac{50}{x}+8\right) = (x+4)\left(\frac{50+8x}{x}\right) = \frac{8x^2+82x+200}{x}.$$

$$\begin{aligned}\frac{dA}{dx} &= (8x^2+82x+200)\times(-x^{-2}) + (x^{-1})\times(16x+82)\\ &= -\frac{8x^2+82x+200}{x^2} + \frac{16x+82}{x}\\ &= \frac{16x^2+82x-(8x^2+82x+200)}{x^2}\\ &= \frac{8x^2-200}{x^2}.\end{aligned}$$

$\frac{dA}{dx} = 0$ when $8x^2 = 200$ ie when $x = \pm 5$. Clearly $x = -5$ cannot be a solution. The table below tells us that we have a minimum when $x = 5$.

x	< 5	5	> 5
A'	$-ve$	0	$+ve$
A	$\searrow$	162	$\nearrow$

When $x = 5$, $y = 10$ and the overall area is 162 cm^2, so the overall dimensions that make the area of the advertisement a minimum are 9 cms by 18 cms.

14. ii

The function $y = x^2e^x$ is positive for all values of $x \neq 0$, so the graph of the function is above the x-axis for all values of $x \neq 0$ and touches it at $(0,0)$. Differentiating we get,

$$\frac{dy}{dx} = x^2e^x + e^x(2x) = xe^x(x+2).$$

This is equal to zero when $x = 0$ or $x = -2$, and so the stationary points are $(-2, 4e^{-2})$ and $(0,0)$.

x	< -2	-2	$> -2, < 0$	0	> 0
y'	$+ve$	0	$-ve$	0	$+ve$
y	$\nearrow$	$4e^{-2}$	$\searrow$	0	$\nearrow$

The table indicates that we have a maximum at $(-2, 4e^{-2})$ and a minimum at $(0,0)$.

$$\frac{d^2y}{dx^2} = x^2e^x + e^x(2x) + 2xe^x + e^x(2) = e^x(x^2 + 4x + 2).$$

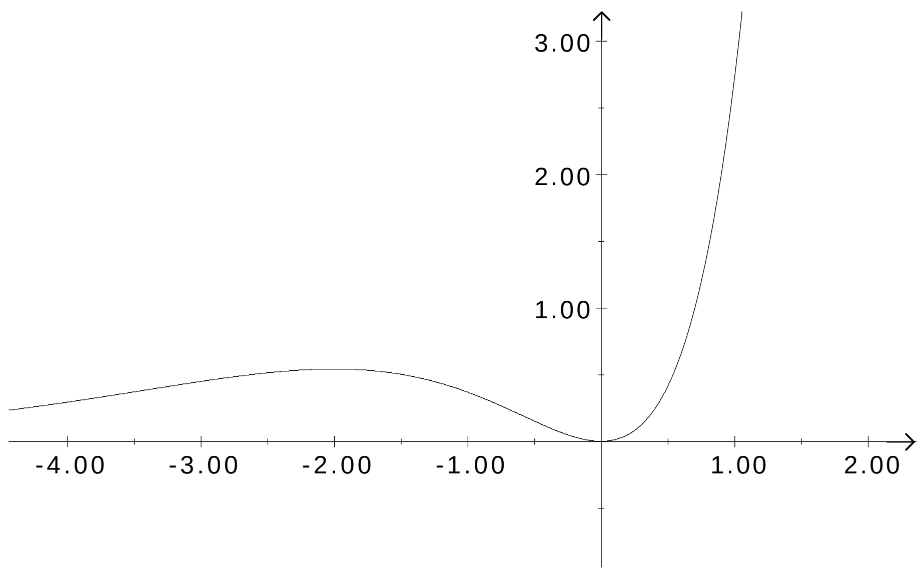
$\dfrac{d^2y}{dx^2} = 0$ when $x^2 + 4x + 2 = 0$. We solve this using the quadratic formula.

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{(4)^2 - 4(1)(2)}}{2} \\ &= \frac{-4 \pm \sqrt{8}}{2} \\ &= \frac{-4 \pm 2\sqrt{2}}{2} \\ &= -2 \pm \sqrt{2}. \end{aligned}$$

We have possible inflection points when $x = -2 - \sqrt{2}$ and $x = -2 + \sqrt{2}$, which we confirm in the following table.

x	$< -2 - \sqrt{2}$	$-2 - \sqrt{2}$	$> -2 - \sqrt{2}, < -2 + \sqrt{2}$	$-2 + \sqrt{2}$	$> -2 + \sqrt{2}$
y''	$+ve$	0	$-ve$	0	$+ve$
y	concave up	0.38	concave down	0.19	concave up

We now have enough information to sketch the curve.



16. ii

$$\begin{aligned}\log\left(\frac{10^x}{100^x}\right) &= \log(10^x) - \log(100^x)\\ &= x\log 10 - x\log 100\\ &= x - x\log(10^2)\\ &= x - 2x\log 10\\ &= x - 2x\\ &= -x.\end{aligned}$$

17. ii

$$\begin{aligned}\ln(x^2+1) &= 3\\ e^{\ln(x^2+1)} &= e^3\\ x^2+1 &= e^3\\ x^2 &= e^3-1\\ x &= \pm\sqrt{e^3-1}\\ &= \pm 4.37.\end{aligned}$$

Both of these solutions are valid as $(\pm 4.37)^2 + 1 > 0$.

19. The population at $t = 0$ is $P = Ae^0 = A$. The population doubles after 5 years so when $t = 5$, $P = 2A$.

i When $t = 5$, $P = 2A$ so, $2A = Ae^{5k}$ ie $e^{5k} = 2$. Therefore, $k = \dfrac{\ln 2}{5}$.

ii When $t = 20$ $P = Ae^{\frac{\ln 2}{5}\cdot 20} = Ae^{4\ln 2} = Ae^{\ln(2^4)} = 16A$.

iii We need to find the value of t when $P = 4A$. So

$$4A = Ae^{\frac{\ln 2}{5}t}$$

$$\begin{aligned}
e^{\frac{\ln 2}{5}t} &= 4 \\
\frac{\ln 2}{5}t &= \ln 4 \\
t &= \frac{5\ln 4}{\ln 2} \\
&= 10.
\end{aligned}$$

Therefore, the population is four times its initial value after 10 years.

21. ii $\sin u = 1$ when $u = -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$ etc.

$\sin\left(3x - \dfrac{\pi}{2}\right) = 1$ when $\left(3x - \dfrac{\pi}{2}\right) = -\dfrac{3\pi}{2}$ or $\left(3x - \dfrac{\pi}{2}\right) = \dfrac{\pi}{2}$ or $\left(3x - \dfrac{\pi}{2}\right) = \dfrac{5\pi}{2}$ or $\left(3x - \dfrac{\pi}{2}\right) = \dfrac{7\pi}{2}$ etc.

That is, $\sin\left(3x - \dfrac{\pi}{2}\right) = 1$ when $x = -\frac{\pi}{3}$ or $x = \frac{\pi}{3}$ or $x = \pi$ or $x = \frac{5\pi}{3}$ etc.

So, the values of x between 0 and 2π for which $\sin\left(3x - \dfrac{\pi}{2}\right) = 1$ are $x = \frac{\pi}{3}$, $x = \pi$ and $x = \frac{5\pi}{3}$.

23. i When $d = 300$, $I = 0.3I(0)$ so,

$$\begin{aligned}
0.3I(0) &= I(0)e^{-300k} \\
e^{-300k} &= 0.3 \\
-300k &= \ln 0.3 \\
k &= -\frac{\ln 0.3}{300} \\
&= 0.004.
\end{aligned}$$

ii We want to find d when $I(d) = 0.5I(0)$ so,

$$\begin{aligned}
0.5I(0) &= I(0)e^{-0.004d} \\
e^{-0.004d} &= 0.5 \\
-0.004d &= \ln 0.5 \\
d &= -\frac{\ln 0.5}{0.004} \\
&= 173.3.
\end{aligned}$$

Therefore, the intensity of the sunlight would be decreased by half at a depth of 173 units below the surface.