

Mathematics Learning Centre



The University of Sydney

# Curving sketching using calculus

Jackie Nicholas

# 1 Curve sketching using calculus

## 1.1 Some General strategies for graphing polynomials

The following steps may be helpful in sketching a general polynomial.

1. Find the intercept on the  $Y$  axis (not because it's particularly illuminating but because it's easy).
2. Note the degree of the polynomial. There are four possibilities:
  - i. If the degree is an even number and the coefficient of the dominant term is positive, then for large positive and large negative values of  $x$ ,  $y$  will be large and positive.
  - ii. If the degree is even and the coefficient of the dominant term is negative, the value of  $y$  will be large and negative for large values of  $x$  (both positive and negative).
  - iii. If the degree of the polynomial is odd and the coefficient of the dominant term is positive, then for large positive and large negative values of  $x$ , the value of  $y$  will be large positive and large negative respectively.
  - iv. If the degree of the polynomial is odd and the coefficient of the dominant term is negative, then for large positive and large negative values of  $x$ , the value of  $y$  will be large negative and large positive respectively.
3. Investigate where the curve is increasing and decreasing and whether it has any stationary points.
4. Find the points of inflection if desired.
5. Find the intercept(s) on the  $X$  axis if it is easy to do so.

**Remember you don't always need all these steps — calculate as much as you need to get an idea of the shape. If a step is very difficult or laborious, leave it out.**

We will illustrate the above ideas with some examples.

### Example:

Sketch the graph of  $y = x^3 + x + 1$ .

### Solution:

1. Find the intercept on the  $Y$  axis by putting  $x = 0$ . This gives  $y = 1$ . At least we now have one point on the graph.
2. The degree is 3. Since the degree of the dominant term ( $x^3$ ) is odd and its coefficient positive, we know that for large positive and large negative values of  $x$ , the value of  $y$  will be large positive and large negative respectively. (Compute values for, say,  $x = 100$  and  $-100$ .)

So, from what we've learnt so far, the curve must look something like Figure 1.

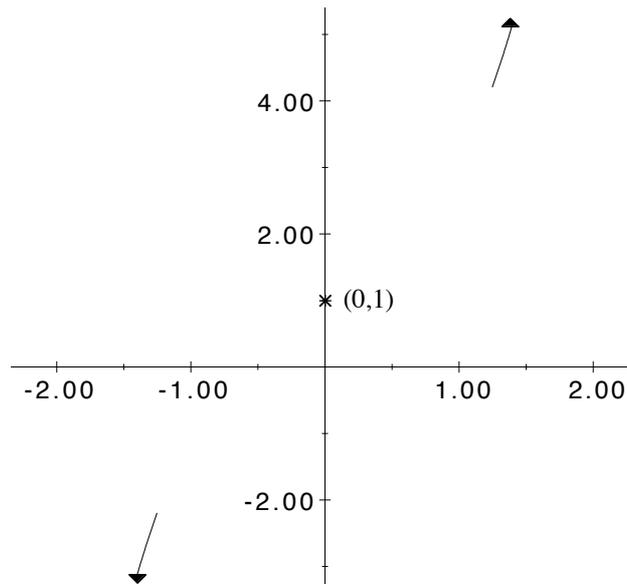


Figure 1: Illustration of information so far for  $y = x^3 + x + 1$ .

**3.** At this point we will look at the derivative of  $x^3 + x + 1$  to determine the stationary points (if any) and the intervals in which the curve increases or decreases.

Now  $\frac{dy}{dx} = 3x^2 + 1$ . This is always positive, which tells us that the curve is increasing everywhere.

Therefore, there are no stationary points.

**4.** We can find any points of inflection by looking at the second derivative:

$$\frac{d^2y}{dx^2} = 6x = 0$$

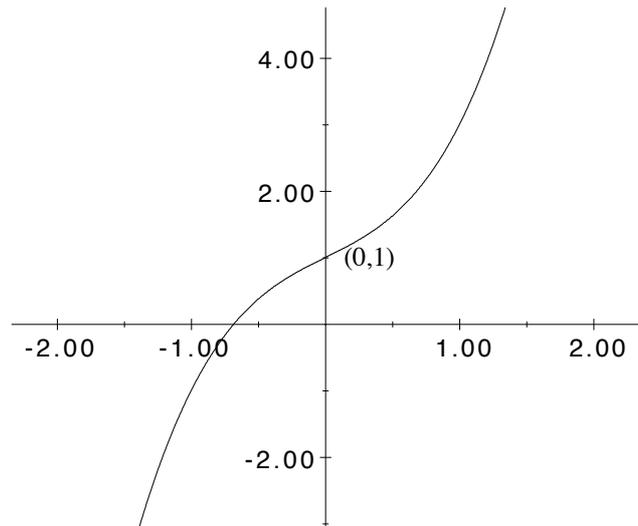
at  $x = 0$ .

For  $x < 0$ ,  $\frac{d^2y}{dx^2} < 0$ , so the curve is concave down; and for  $x > 0$ ,  $\frac{d^2y}{dx^2} > 0$ , so the curve is concave up.

Accordingly, there is a point of inflection (a change in direction of the bend of the curve) at  $(0, 1)$ .

**5.** We would not normally bother with the  $X$  intercepts unless we need to know them. Such knowledge does not, in the present case, add much to our picture of the curve. Since  $y = -1$  when  $x = -1$  and  $y = 1$ , when  $x = 0$ , there must be a root between  $-1$  and  $0$ . A better approximation of the location of this root can be found using numerical methods.

The graph of  $y = x^3 + x + 1$  is given in Figure 2.

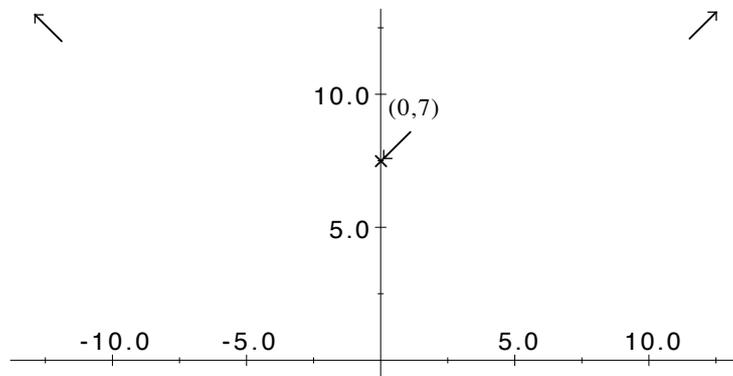
Figure 2: Graph of  $y = x^3 + x + 1$ .

**To summarise**, note that we worked out the shape of the curve with almost no calculation of values of the function. (The opposite of a computer graphing program.) Not only was our deductive method faster than laborious computation, it offered far more insight into the behaviour of the function and the shape of the graph. Also we used calculus judiciously. Calculus plays a much smaller part in curve sketching than is commonly believed; it is just one of the tools at our disposal.

**Example:** Sketch the graph of  $y = x^4 - 2x^2 + 7$ .

**Solution:**

1. The  $Y$  intercept is readily found to be  $(0, 7)$ .
2. If  $x$  is large positive then  $y$  is large positive. If  $x$  is large negative then  $y$  is large positive. Notice that the curve will be symmetrical about the  $Y$  axis because  $x$  appears only as even powers. That is,  $f(x) = f(-x)$ . This symmetry also implies, in this case, that the curve is horizontal at  $(0, 7)$ . Otherwise there would have to be a sharp point (called a cusp) there, and polynomials don't have cusps.

Figure 3: Knowledge so far of  $y = x^4 - 2x^2 + 7$ .

**3.** The derivative of a quartic is a cubic and can have at most three roots. So there can be at most three stationary points to a quartic. We can find the stationary points from the derivative and draw up a table which summarises the changes of direction of the curve.

$$\frac{dy}{dx} = 4x^3 - 4x.$$

Thus  $\frac{dy}{dx} = 0$  when  $4x^3 - 4x = 0$  i.e. when  $x^3 - x = 0$ .

Factorising:  $x(x^2 - 1) = 0$  so

$$x(x - 1)(x + 1) = 0$$

i.e.  $x = -1$ ,  $x = 0$ , or  $x = 1$ .

The stationary points are at  $(-1, 6)$ ,  $(0, 7)$  and  $(1, 6)$

A change can only occur at the stationary points, so we choose values on either side of these points which are  $x = -1$ ,  $x = 0$ ,  $x = 1$ , and evaluate  $\frac{dy}{dx}$  there. We can also sketch an ‘attitude’ diagram from the table showing the direction of the curve.

From the table and ‘attitude’ diagram below we see there is a local minimum at  $(-1, 6)$  and  $(1, 6)$ , and a local maximum at  $(0, 7)$ .

$x$	-2	-1	-0.5	0	0.5	1	2
$\frac{dy}{dx}$	-24	0	1.5	0	-1.5	0	24
curve	decreases	horiz.	increases	horiz.	decreases	horiz.	increases
attitude	$\searrow$	$\rightarrow$	$\nearrow$	$\rightarrow$	$\searrow$	$\rightarrow$	$\nearrow$

(Remember: if  $\frac{dy}{dx} < 0$  the function is decreasing, if  $\frac{dy}{dx} > 0$  the function is increasing, and if  $\frac{dy}{dx} = 0$  the function is locally stationary.)

We note from the position of the local minima that the curve never cuts the  $X$  axis. The graph of  $y = x^4 - 2x^2 + 7$  is shown in Figure 4.

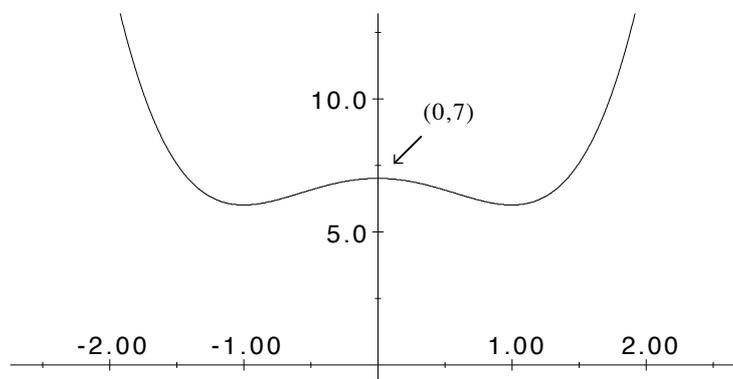


Figure 4: Graph of  $y = x^4 - 2x^2 + 7$ .

### Exercise 1.1

1. a. Estimate where the points of inflection are in the preceding example.

- b. Check your estimate using the second derivative.
- 2. Sketch the graph of  $y = (x - 1)^4 + 8x$ .
- 3. Sketch the graph of  $y = -3x^5 - 10x^3 - 18x + 31$ . Find any turning points and points of inflection.

## 1.2 Strategies for graphing functions in general

We can slightly modify our strategies for graphing polynomials to more general functions. Instead of looking at the degree and sign of the dominant term of the polynomial, we will look generally at the function we are given and see what can be deduced about its graph. We will look at the behaviour of the function as  $x \rightarrow \infty$  (ie large positive values) and as  $x \rightarrow -\infty$  (ie large negative values).

Let's illustrate this with an example.

**Example:** Sketch the graph of the function  $y = x^2e^x$ .

**Solution:**

- 1. The  $Y$  intercept is 0, since  $y = 0^2e^0 = 0$ . (Note  $e^0 = 1$ .)
- 2. The function  $y = e^x$  is positive for all values of  $x$  and so,  $y = x^2e^x \geq 0$  for all values of  $x$ . Therefore, the graph of  $y = x^2e^x$  is in the upper half plane for  $x \neq 0$  and touches the  $X$  axis at  $x = 0$ .

Also, as  $x \rightarrow \infty$ ,  $y \rightarrow \infty$  and as  $x \rightarrow -\infty$ ,  $y \rightarrow 0$ .

- 3. Differentiating we have:

$$\begin{aligned} \frac{dy}{dx} &= x^2e^x + e^x \cdot 2x \\ &= xe^x(x + 2) \\ &= 0 \quad \text{when } x = 0 \text{ or } x = -2. \end{aligned}$$

(Note that  $e^x > 0$  for all  $x$ .)

The points  $(0, 0)$  and  $(-2, 4e^{-2})$  are stationary points.

From the table and 'attitude' diagram below we see there is a local maximum at  $(-2, 4e^{-2})$  and a local minimum at  $(0, 0)$ .

$x$	$< -2$	$-2$	$> -2, < 0$	$0$	$0$
$\frac{dy}{dx}$	+ve	0	-ve	0	+ve
curve	increases	horiz.	decreases	horiz.	increases
attitude					

We can now put all this information together and sketch the curve. The graph of  $y = x^2e^x$  is given in Figure 5.

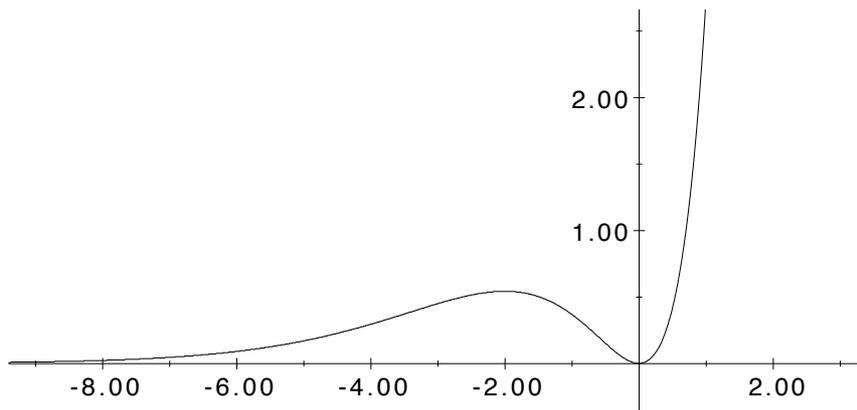


Figure 5: Graph of  $y = x^2e^x$ .

Notice how the curve is asymptotic to the  $X$  axis as  $x \rightarrow -\infty$ . We deduced this from **2.** above as when  $x \rightarrow -\infty$   $y \rightarrow 0$ , and  $y > 0$  for all  $x \neq 0$ .

We can see from our sketch that there are two points of inflection. We could find these if we liked by examining the second derivative but in this case it is a lot of effort for not much more information.

### Exercise 1.2

1. Sketch the graph of  $y = te^t$ .
2. Sketch the graph of  $y = \frac{1}{x^2+3}$ .

## 1.3 Solutions to exercises

### Solutions to exercise 1.1

1. **a.** By inspection of the graph in Figure 4, there are two points of inflection about when  $x = -0.5$  and when  $x = 0.5$ .
- b.** Differentiating the first derivative we get:

$$\begin{aligned} \frac{d^2x}{dx^2} &= 12x^2 - 4 \\ &= 4(3x^2 - 1) \\ &= 0 \quad \text{when } x^2 = \frac{1}{3}, \text{ ie when } x = \pm \frac{1}{\sqrt{3}}. \end{aligned}$$

So,  $x \approx 0.58$  or  $x \approx -0.58$ .

2. When  $x = 0$   $y = (0 - 1)^4 + 8(0) = 1$ .

The dominant term is  $x^4$ . So for large positive and large negative values of  $x$ ,  $y$  is large and positive.

$$\frac{dy}{dx} = 4(x - 1)^3 + 8 = 0 \text{ when } (x - 1)^3 = -2$$

ie when  $x = 1 - \sqrt[3]{2}$ . There is a stationary point at about  $(-0.26, 0.44)$ .

$x$	$-1$	$1 - \sqrt[3]{2}$	$0$
$\frac{dy}{dx}$	$-24$	$0$	$4$
curve	decreases	horiz.	increases
attitude	$\searrow$	$\rightarrow$	$\nearrow$

There is a local minimum when  $x = 1 - \sqrt[3]{2}$ , ie at about  $(-0.26, 0.44)$ .

$\frac{d^2y}{dx^2} = 12(x - 1)^2$  and so  $\frac{d^2y}{dx^2} > 0$  for all values of  $x \neq 1$ . So, the curve must be concave up everywhere. There are no points of inflection.

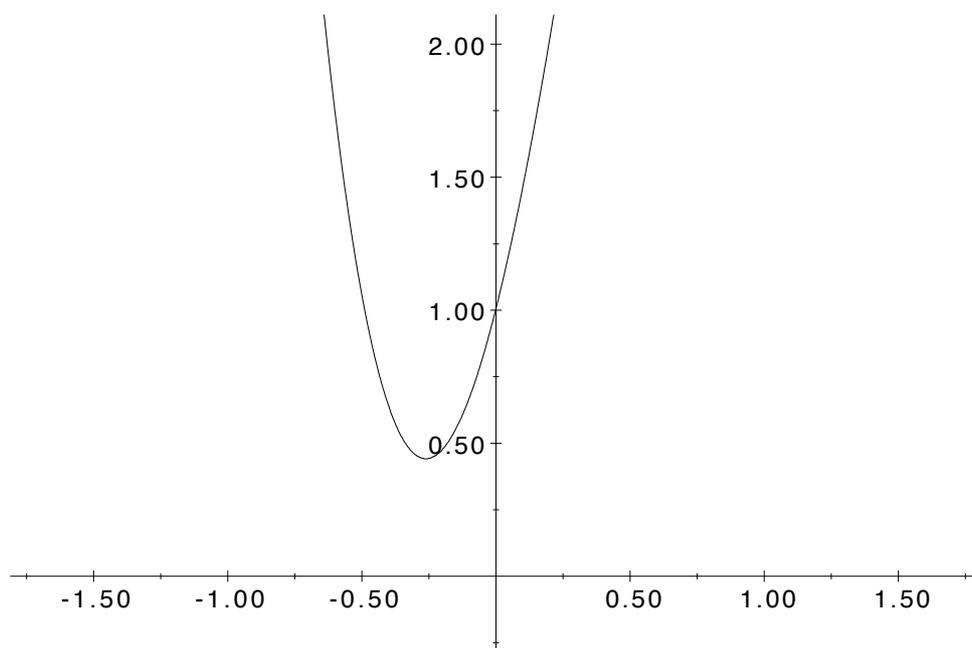


Figure 6: Graph of  $y = (x - 1)^4 + 8x$ .

3. The graph of  $y = -3x^5 - 10x^3 - 18x + 31$  passes through the point  $(0, 31)$ .

When  $x$  is large positive,  $y$  is large negative. When  $x$  is large negative,  $y$  is large positive.

$\frac{dy}{dx} = -15x^4 - 30x^2 - 18$  which is negative everywhere, so the curve is decreasing everywhere (no stationary points).

Looking at the second derivative:  $\frac{d^2y}{dx^2} = -60x^3 - 60x = -60x(x^2 + 1)$ .

This is positive when  $x < 0$  and negative when  $x > 0$ . Hence the curve is concave up to the left of  $(0, 31)$  and concave down to the right of  $(0, 31)$ . The graph of  $y = -3x^5 - 10x^3 - 18x + 31$  is given in Figure 7.

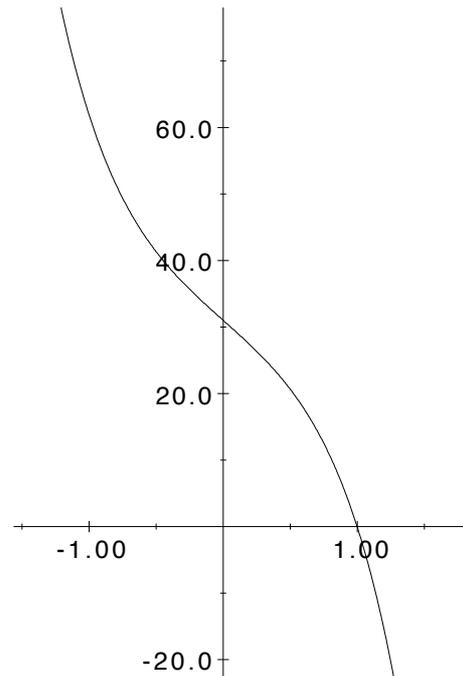


Figure 7: Graph of  $y = -3x^5 - 10x^3 - 18x + 31$ .

**Solutions to exercise 1.2**

1. When  $t = 0$ ,  $y = te^t = 0e^0 = 0$ . The  $Y$  intercept is 0.

$y = te^t$  is defined for all values of  $t$ .  $e^t > 0$  for all values of  $t$ , so when  $t < 0$ ,  $y = te^t < 0$  and when  $t > 0$ ,  $y = te^t > 0$ . These regions in the plane are marked in Figure 8.

As  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ . As  $x \rightarrow -\infty$ ,  $y \rightarrow 0$ .

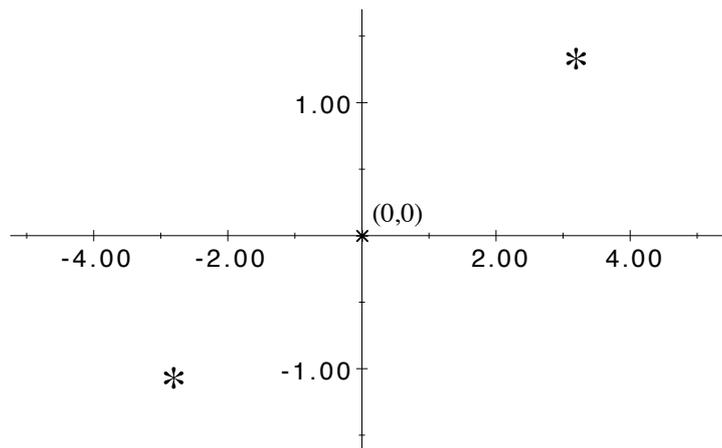


Figure 8: Graph lies in the regions marked by an asterisk.

Differentiating we get:

$$\begin{aligned} \frac{dy}{dt} &= t.e^t + e^t.(1) \\ &= e^t(t + 1). \end{aligned}$$

So,  $\frac{dy}{dt} = 0$  when  $t = -1$ . There is a stationary point at  $(-1, -e^{-1})$ .

$x$	$< -1$	$-1$	$> -1$
$\frac{dy}{dx}$	-ve	0	+ve
curve	decreases	horiz.	increases
attitude	$\searrow$	$\rightarrow$	$\nearrow$

Therefore, there is a local minimum at  $(-1, -e^{-1})$ .

To find any points of inflection we will need the second derivative.

$$\begin{aligned} \frac{d^2y}{dt^2} &= e^t \cdot 1 + (t + 1) \cdot e^t \\ &= e^t(t + 2) \\ &= 0 \quad \text{when } t = -2. \end{aligned}$$

So, there is a **possible** point of inflection at  $(-2, -2e^{-2})$ .

To say if there is a point of inflection here for sure, we need to check that the concavity of the curve changes.

For  $t < -2$ ,  $\frac{d^2y}{dt^2} < 0$  so the curve is concave down for  $t < -2$ . For  $t > -2$ ,  $\frac{d^2y}{dt^2} > 0$  so the curve is concave up for  $t > -2$ . The change in concavity at  $(-2, -2e^{-2})$  tells us that there is a point of inflection here.

A sketch of the function  $y = te^t$  is given in Figure 9.

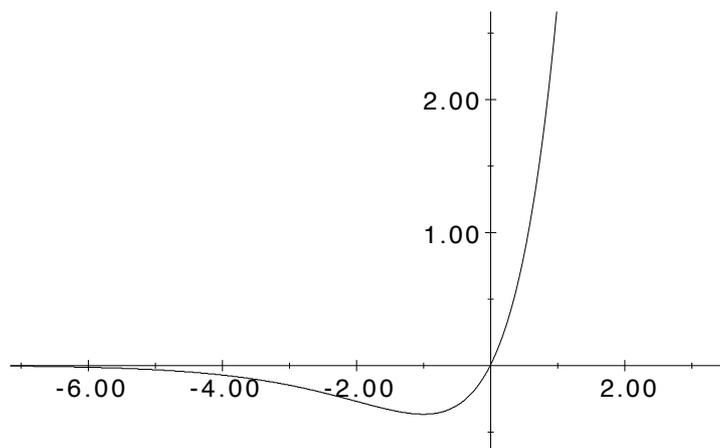


Figure 9: Graph of  $y = te^t$ .

- When  $x = 0$ ,  $y = \frac{1}{3}$  so the point  $(0, \frac{1}{3})$  lies on the curve.

The function  $y = \frac{1}{x^2 + 3}$  is defined for all values of  $x$  as the denominator is always positive. Therefore,  $y = \frac{1}{x^2 + 3} > 0$  for all values of  $x$  and its graph is in the upper half plane.

$y = \frac{1}{x^2 + 3}$  is symmetrical about the  $Y$  axis, as  $f(-x) = f(x)$ .

Also, when  $x \rightarrow \infty$ ,  $y \rightarrow 0$  and when  $x \rightarrow -\infty$ ,  $y \rightarrow 0$  since the denominator is positive and grows as  $x$  grows in magnitude.

Differentiating using the quotient rule we get:

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x^2 + 3)(0) - (1)(2x)}{(x^2 + 3)^2} \\ &= \frac{-2x}{(x^2 + 3)^2} \\ &= 0 \quad \text{when } x = 0. \text{ (Note the denominator is always positive.)} \end{aligned}$$

So, there is a stationary point at  $(0, \frac{1}{3})$ .

$x$	$< 0$	$0$	$> 0$
$\frac{dy}{dx}$	+ve	0	-ve
curve	increases	horiz.	decreases
attitude	$\nearrow$	$\rightarrow$	$\searrow$

The point  $(0, \frac{1}{3})$  is a local maximum.

A sketch of the graph of  $y = \frac{1}{x^2 + 3}$  is given in Figure 10.

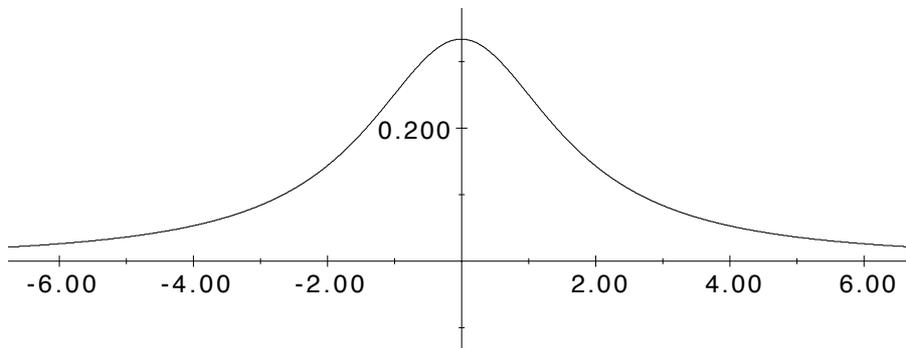


Figure 10: Graph of  $y = \frac{1}{x^2 + 3}$ .