Solving linear equations by Gaussian elimination

Jackie Nicholas
Mathematics Learning Centre
University of Sydney

The big idea

The big idea is to progressively replace our original system of linear with an equivalent system of linear equations which has the same solution set.

We do this by using elementary row operations to systematically simplify the augmented matrix representing our system of linear equations.

By using only elementary row operations, we do not lose any information contained in the augmented matrix.

Our strategy is to progressively alter the augmented matrix using elementary row operations until it is in **row echelon form**.

This process is known as Gaussian elimination.

Row echelon form

A matrix is in row echelon form if

1. the first nonzero entries of rows are equal to 1
2. the first nonzero entries of consecutive rows appear to the right
3. rows of zeros appear at the bottom.

Here are some examples of matrices in row echelon form.

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & -1 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & -3 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Reducing the augmented matrix

Consider the augmented matrix $\left[\begin{array}{rrr|r} 3 & 4 & -1 & 1 \\ 1 & -1 & 1 & 3 \\ -1 & 2 & 3 & 5 \end{array}\right]$.

Our first step is to get a 1 in the top left of the matrix by using an elementary row operation:

$$R_1 \longleftrightarrow R_2 \begin{bmatrix} \color{red}{1} & -1 & 1 & 3 \\ 3 & 4 & -1 & 1 \\ -1 & 2 & 3 & 5 \end{bmatrix}.$$

Next we use this 1, called the leading 1 (in red), to eliminate the nonzero entries below it:

$$R_2 - 3R_1 \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 7 & -4 & -8 \\ -1 & 2 & 3 & 5 \end{bmatrix} R_3 + R_1 \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 7 & -4 & -8 \\ 0 & 1 & 4 & 8 \end{bmatrix}.$$

Reducing the matrix

We now move down and across the matrix to get a leading 1 in the $(2,2)$ position (highlighted in red): $\begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 7 & -4 & -8 \\ 0 & 1 & 4 & 8 \end{bmatrix}$.

We can do this by $R_3 \longleftrightarrow R_2$ $\begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 4 & 8 \\ 0 & 7 & -4 & -8 \end{bmatrix}$.

Next, we use this leading 1 (in red) to eliminate all the nonzero entries below it:

$$R_3 - 7R_2 \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & -32 & -64 \end{bmatrix}.$$

Reducing the matrix

Finally, we move down and across to the $(3, 3)$ position

(highlighted red) $\begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & \color{red}{-32} & -64 \end{bmatrix}$.

We make this entry into a leading 1:

$$-\frac{1}{32}R_3 \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & 1 & \frac{-64}{-32} \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Our matrix is now in row echelon form.

Solving by back substitution

$$\begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

With our matrix now in row echelon form, we can solve our system of linear equations by back substitution.

Reading off from the last row: $z = 2$.

From the second row: $y + 4z = 8$, ie $y = 8 - 4z = 8 - 4(2) = 0$.

From the first row: $x - y + z = 3$, ie $x = 3 + y - z = 3 + 0 - 2 = 1$.

So the solution is: $x = 1$, $y = 0$, $z = 2$.