Mathematics Learning Centre

The University of Sydney

Exponential Functions

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1 Exponential Functions

1.1 The functions $y = 2^x$ and $y = 2^{-x}$

From our work on exponents we know that, if b is a positive number, we can make sense of the expression b^x for all real numbers x. It turns out that functions of the type $y = f(x) = b^x$, where b is a positive number, are of great importance in mathematics and in all branches of the sciences.

To get an indication of how these functions behave we have graphed the function $f(x) = 2^x$ in Figure 1.

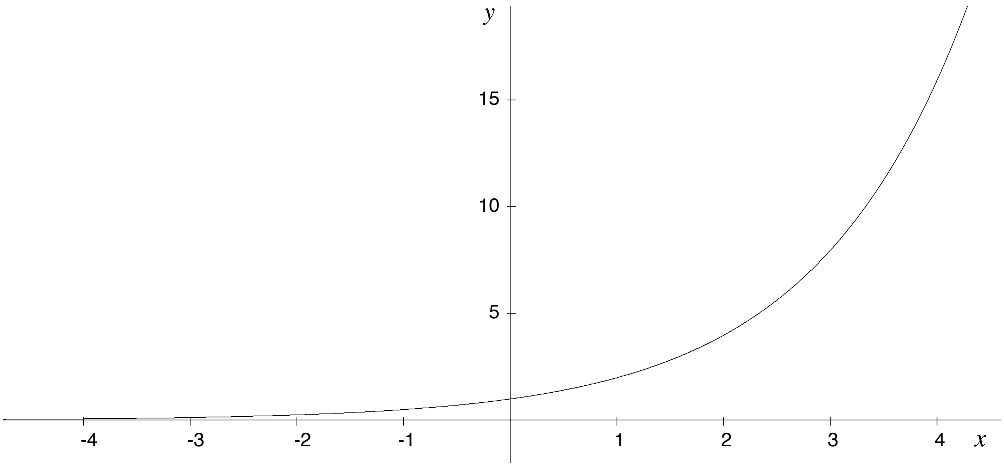

Figure 1: Graph of the function $f(x) = 2^x$

You should be aware of several important features of this graph.

The function $f(x) = 2^x$ is always positive (the graph of the function never cuts the x-axis), although the value of the function gets very close to zero for values of x very large negative (ie a long way to the left along the x-axis). For example, when $x = -5$ we have $2^x = 0.03125$.

The function 2^x increases very rapidly for large values of x. From the rules of exponents, we know that $2^{x+1} = 2 \times 2^x$. In words, the value of 2^x doubles if x is increased by 1.

The graph of $y = 2^x$ intercepts the y-axis at $y = 1$. We should expect this because we know from the rules of exponents that $2^0 = 1$.

Figure 2 displays the graph of the function $f(x) = 2^{-x}$.

How is the graph of $y = 2^{-x}$ related to the graph of $y = 2^x$? Well, if we set $x = 1$ then $2^{-x} = 2^{-1} = \frac{1}{2}$, which is the value which would have been obtained by setting $x = -1$ in the function $y = 2^x$. In the same way, if we set $x = -7$ in the function $y = 2^{-x}$ then we obtain the same value as we would by setting $x = 7$ in the function $y = 2^x$. Proceeding like this we see that *the graph of the function $y = 2^{-x}$ is the reflection in the y*-axis *of the graph of $y = 2^x$*. Compare Figure 1 with Figure 2.

From the rules of exponents, it follows that $2^{-x} = (2^{-1})^x = (\frac{1}{2})^x$. The function $y = 2^{-x}$ is the same as the function $y = (\frac{1}{2})^x$, and so

$$2^{-(x+1)} = (\frac{1}{2})^{x+1} = \frac{1}{2} \times (\frac{1}{2})^x = (\frac{1}{2}) \times 2^{-x}.$$

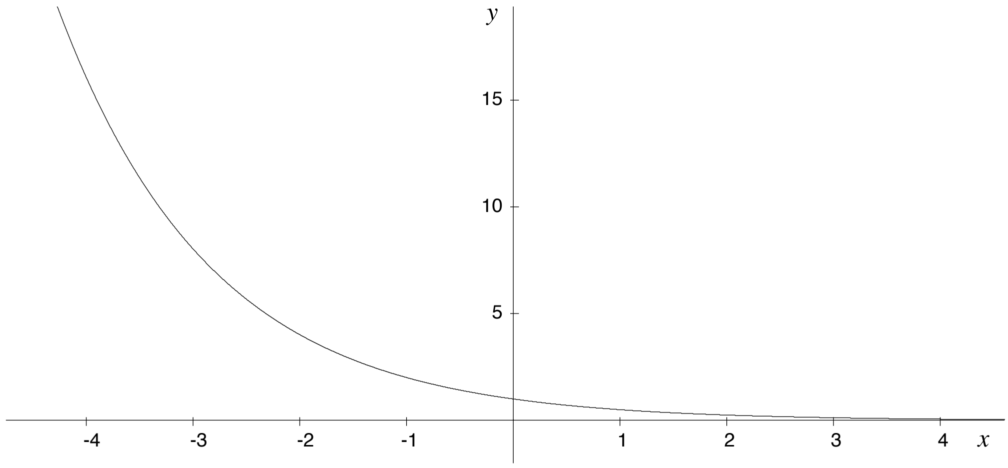

Figure 2: Graph of the function $y = 2^{-x}$

In words, the value of the function $y = 2^{-x}$ is decreased by a factor of $\frac{1}{2}$ if x is increased by 1.

1.2 The functions $y = b^x$ and $y = b^{-x}$

Any function of the form $y = b^x$ where $b > 0$ and $b \neq 1$ behaves like one of the functions $y = 2^x$ or $y = (\frac{1}{2})^x = 2^{-x}$.

If $b > 1$ then the function $y = b^x$ is increasing and behaves like $y = 2^x$.

If $b < 1$ then the function is decreasing and behaves like $y = (\frac{1}{2})^x = 2^{-x}$.

If $b = 1$ then $y = 1^x = 1$ for all x. Notice that regardless of the value of b, providing always that $b > 0$, the function $y = b^x$ intercepts the y-axis at $y = 1$. This is because $b^0 = 1$ for all numbers b. Figure 3 shows the graphs of the functions $y = b^x$ for various values of b.

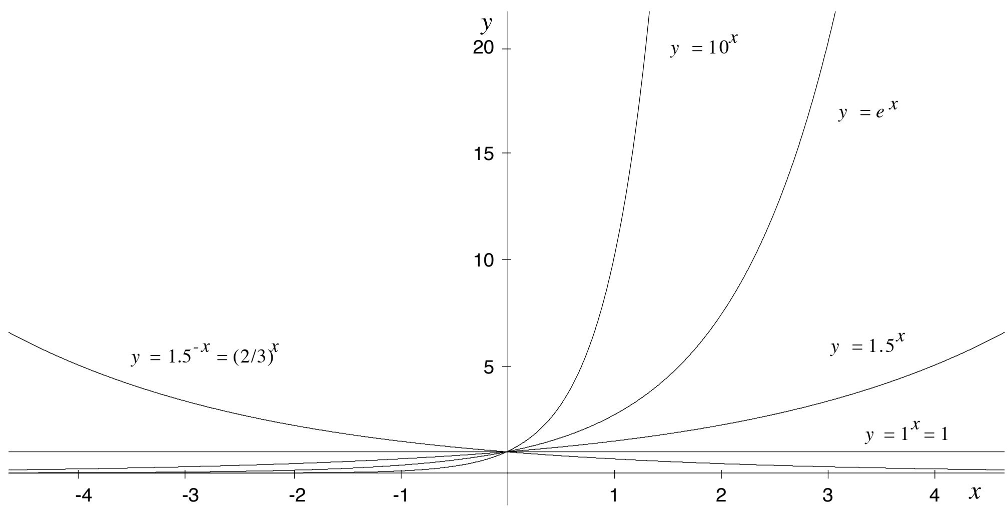

Figure 3: Graphs of $y = b^x$ for various values of b.

1.3 The functions $y = e^x$ and $y = e^{-x}$

There is a number called e which has a special importance in mathematics. Like the number π, the number e is an *irrational* number, which is equivalent to saying that it has a non-terminating, non-repeating decimal representation. In other words we can never write down exactly what e is. To 5 decimal places it is equal to 2.71828, but this is just an approximation of the correct value. Unless you really need to write down an approximate value for e it is more convenient and accurate to leave the symbol e in expressions involving this number. For example, it is preferable to write $2e$ rather than 2×2.71828 or 5.43656.

In mathematics the functions e^x and e^{-x} are particularly important. Because of this we have graphed them in Figure 4. You can see how similar these functions are to the other exponential functions.

The function $y = e^x$ is often referred to as *the* exponential function, and is even given another special symbol, exp, so that $\exp(x) = e^x$ and $\exp(-x) = e^{-x}$.

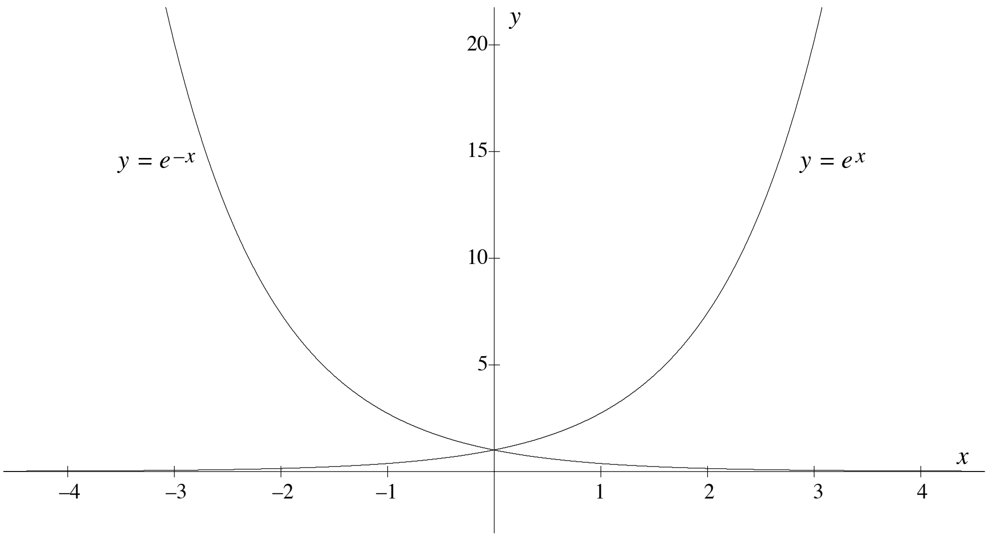

Figure 4: Graphs of e^x and e^{-x}.

1.4 Summary

Functions of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$ are called *exponential functions.*

If $b < 1$ then b^x is a decreasing function, and if $b > 1$ then b^x is an increasing function.

The function b^{-x} is equal to the function $(\frac{1}{b})^x$.

The number $e \approx 2.71828$ and the functions e^x and e^{-x} are of special importance in mathematics. The function e^x is often given the special name exp, so that $\exp(x) = e^x$ and $\exp(-x) = e^{-x}$.

1.5 Exercises

1. Make a careful sketch of the graphs of the functions $y = 2.5^x$ and $y = 5^{-x}$. Indicate where (if at all) these functions intercept the axes.

2. Which of the following functions are increasing and which are decreasing? You should be able to decide without graphing the functions or substituting any values, though you may do so if you wish.

a. $f(x) = 2.7^x$ **b.** $f(x) = (\frac{1}{2.7})^{-x}$ **c.** $f(x) = 3^{-x}$ **d.** $f(x) = 0.22^x$

3. Which of the following functions are increasing and which are decreasing? If you have understood this section fully you will be able to answer this question without graphing the functions or substituting any values.

a. $f(x) = (\frac{5}{3})^x$ **b.** $f(x) = (\frac{5}{3})^{-x}$ **c.** $f(x) = (\frac{3}{5})^{-x}$ **d.** $f(x) = (\frac{3}{5})^x$

4. Sketch the graphs of the functions $f(x) = 3^x$ and $f(x) = 3^{-x}$. On the same diagrams mark in roughly the graphs of $f(x) = 2.9^x$ and 2.9^{-x}.

5. It is true that $e^{1.09861} \approx 3$. Try it for yourself on a calculator if you have one. How do you think the functions $y = 3^x$ and $y = e^{1.09861x}$ compare? Why? If you cannot solve this otherwise, you might like to try substituting in a few numbers for x in both of the functions and comparing the values.

1.6 Solutions to exercises

1. The graphs of $y = 2.5^x$ and $y = 5^{-x}$ appear below. In both cases the graph intercepts the y-axis at $y = 1$. In neither case does the graph intercept the x-axis, though the graph does get extremely close to the x-axis in both cases.

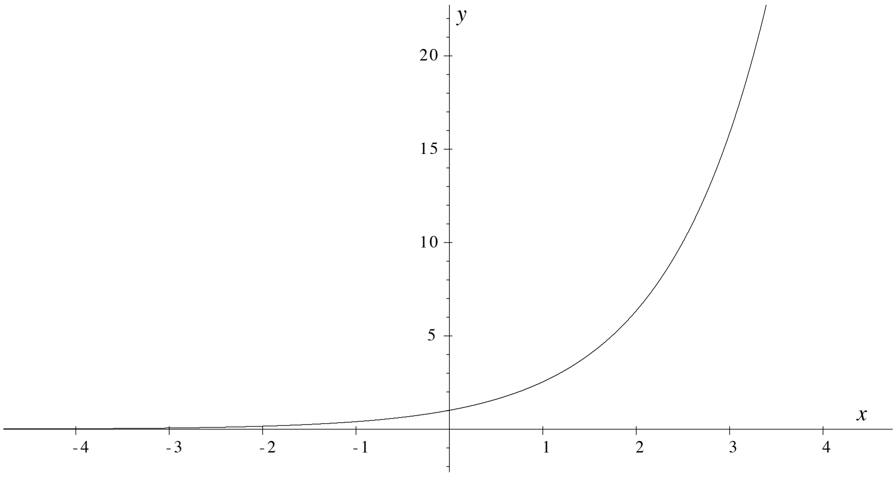

Figure 5: Graph of $y = 2.5^x$.

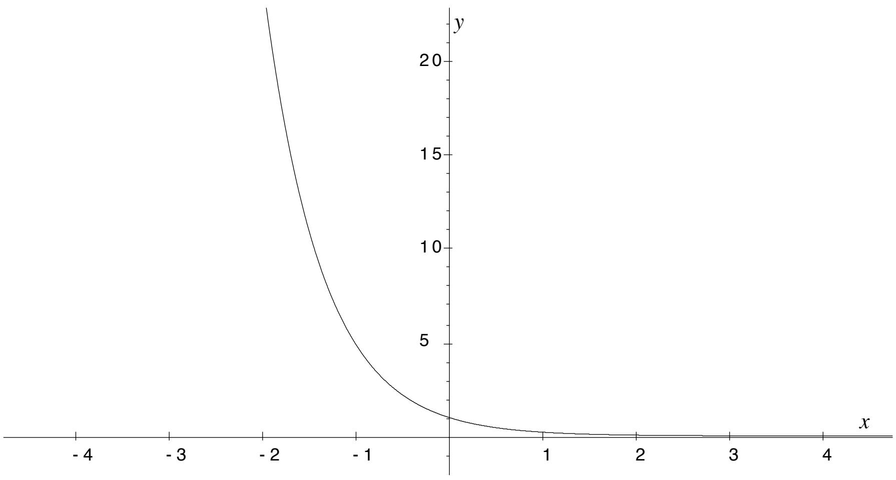

Figure 6: Graph of $y = 5^{-x}$.

2. Remember that the function $f(x) = b^x$ is increasing if $b > 1$ and is decreasing if $b < 1$.

a. $f(x) = 2.7^x$ is increasing since $2.7 > 1$.

b. $f(x) = (\frac{1}{2.7})^{-x} = 2.7^x$, so this function is also increasing.

c. $f(x) = 3^{-x} = (\frac{1}{3})^x$ is decreasing since $\frac{1}{3} < 1$.

d. $f(x) = 0.22^x$ is decreasing since $0.22 < 1$.

3. Again, remember that the function $f(x) = b^x$ is increasing if $b > 1$ and is decreasing if $b < 1$.

a. $f(x) = (\frac{5}{3})^x$ is increasing because $\frac{5}{3} > 1$.

b. $f(x) = (\frac{5}{3})^{-x} = (\frac{3}{5})^x$ is decreasing because $\frac{3}{5} < 1$.

c. $f(x) = (\frac{3}{5})^{-x} = (\frac{5}{3})^x$ is increasing because $\frac{5}{3} > 1$.

d. $f(x) = (\frac{3}{5})^x$ is decreasing because $\frac{3}{5} < 1$.

4. The graphs are drawn in Figures 21 and 22 below. Notice that the graph of $f(x) = 2.9^x$ is very close to the graph of $f(x) = 3^x$, and similarly for the other pair of graphs.

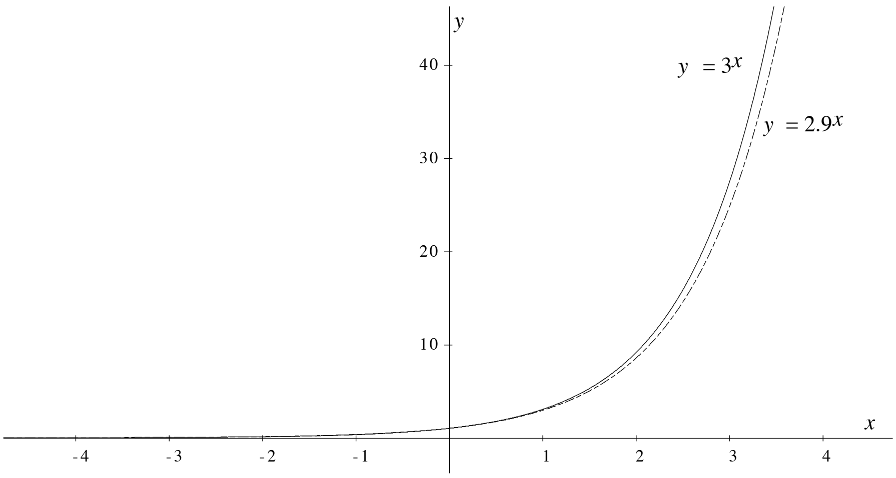

Figure 7: Graphs of $y = 3^x$ and $y = 2.9^x$.

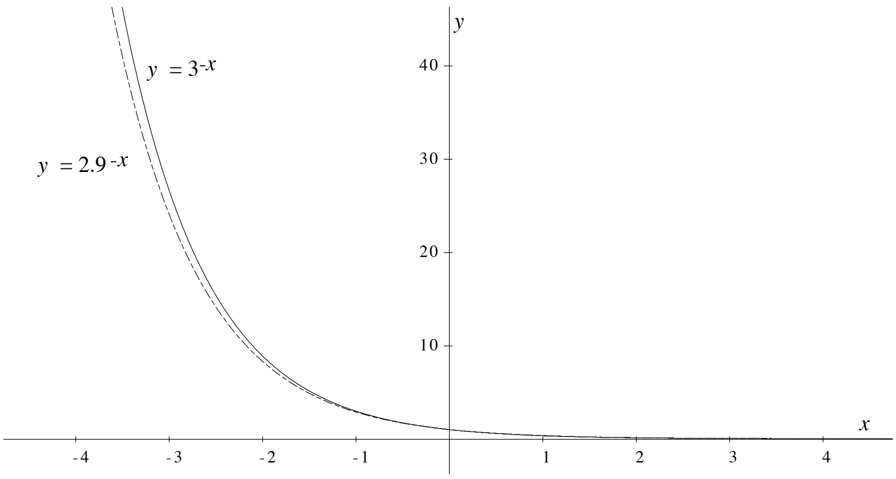

Figure 8: Graphs of $y = 3^{-x}$ and $y = 2.9^{-x}$.

5. On my calculator I get $e^{1.09861} = 2.999993$. Now

$$\begin{aligned} e^{1.09861x} &= (e^{1.09861})^x \\ &\approx 3^x \end{aligned}$$

Thus the functions 3^x and $e^{1.09861x}$ agree very closely with each other.