BRIDGING COURSE IN MATHEMATICS

EXTENSION 1 MATHEMATICS

Exercises and Answers

S. Britton and J. Henderson

School of Mathematics and Statistics

Mathematics Learning Centre

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Chapter 1

Exercises

1.1 Functions I

1. Find the natural domains of the functions defined by:

(i) $f(x) = \dfrac{x}{x^2+2x+1}$ $\quad$ (ii) $f(x) = \dfrac{1}{x} + \dfrac{1}{x+2}$

(iii) $f(t) = \dfrac{t-1}{t^2+1}$ $\quad$ (iv) $f(x) = \sqrt{2x-7}$

(v) $g(x) = \sqrt[3]{2x-7}$ $\quad$ (vi) $f(x) = \sqrt{1-\sqrt{4-x^2}}$

Find the ranges in parts (iv) and (v).

2. In each of the following find an explicit formula for the composite functions $f(g(x))$ and $g(f(x))$. Also find the natural domain of each composite function.

(i) $f(x) = 3x+2$ and $g(x) = -4x-6$

(ii) $f(x) = \dfrac{1}{x}$ and $g(x) = \sqrt{x^2-1}$

(iii) $f(x) = \sqrt{x-1}$ and $g(x) = x^2$

(iv) $f(x) = \dfrac{x^2-1}{x^2+1}$ and $g(x) = \sqrt{x+1}$

(v) $f(x) = x^2-1$ and $g(x) = |x|$

3. Differentiate the following expressions.

(i) (a) x^2 $\quad$ (b) $12x - x^2$ $\quad$ (c) $\dfrac{1}{x^2}$ $\quad$ (d) $\dfrac{3x+1}{2x-3}$

(e) $\sqrt{x}$ $\quad$ (f) $\dfrac{1}{\sqrt{x}}$ $\quad$ (g) $\dfrac{x^2-1}{x}$ $\quad$ (h) $\dfrac{1}{x^3} + \sqrt{x}$

(ii) (a) $(x^2-4)(x^4+3)$ $\quad$ (b) $x^2 \sin x$ $\quad$ (c) $x^2 \tan x$

(d) $e^x(x+3)$ $\quad$ (e) $\dfrac{x+3}{e^x}$ $\quad$ (f) $\dfrac{\sin x}{x}$

(g) $x^3 e^x \sin x$ $\quad$ (h) $\dfrac{1+\tan x}{1-\tan x}$ $\quad$ (i) $\dfrac{1+\sin x}{\cos x}$

(j) $\dfrac{x}{1+\sqrt{x}}$

4. (*i*) Find dy/dx, when $y = 24x + 3x^2 - x^3$. Prove that y has a maximum value of 80 when $x = 4$. When $x = -5$, y again has a value of 80. Explain this.

(*ii*) Write down the gradient of the function $4x^2 + 27/x$. Hence find the value of x for which the function is a maximum or a minimum. Which is it?

(*iii*) Find the equations of the tangent and normal to the curve $y = 2x^2 - 4x + 5$ at $(3, 11)$.

(*iv*) Prove that the curves $y = x^2$, $6y = 7 - x^3$ intersect at right angles at the point $(1, 1)$.

(*v*) It is found that the cost of running a steamer a certain definite distance, at an average speed of V knots, is proportional to

$$V + V^3/100 + 300/V,$$

the first two terms representing the cost of power and the third term the costs, such as wages, which are directly proportional to the time occupied. What is the most economical speed?

(*vi*) Soreau's formulae for the supporting thrust V and the horizontal thrust H of the air on a plane surface making a small angle α with the direction of motion are

$$H = kv^2(a\alpha^2 + b),$$
$$V = kv^2\alpha,$$

where v is the velocity of the plane and k, a and b are constants. For what value of α is the ratio H/V a minimum?

1.2 Functions II

1. Use the chain rule to find $\dfrac{dy}{dx}$ in the following cases.

(*i*) $y = \cos(\sin x)$ (*ii*) $y = \sin(\cos x)$ (*iii*) $y = (\sin x)^3 - (\cos x)^3$

(*iv*) $y = x^2 \sin \dfrac{1}{x}$ (*v*) $y = \dfrac{x}{\sqrt{2+x}}$ (*vi*) $y = \sqrt{x + \sqrt{x}}$

(*vii*) $y = \tan x^2 + \tan^2 x$

2. Given the equations below, find $\dfrac{dy}{dx}$ at the indicated points using implicit differentiation.

(*i*) $x^3 + y^3 = 6xy$ at $(3, 3)$. (*ii*) $y^2 = x^3(2 - x)$ at $(1, 1)$.

(*iii*) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ at $(-3\sqrt{3}, 1)$. (*iv*) $x^2y^2 + xy = 2$ at $(1, 1)$.

3. The height of an isosceles triangle of constant base 10 cm is increasing at the rate of 0.2 cm/sec. How fast is the area increasing? How fast is the perimeter increasing when $h = 10$ cm?

4. A hemispherical tank of radius r metres is filled to the brim with water. As the water evaporates, the volume of water decreases. When the water depth is h metres, the volume of water in the tank is given by

$$V = \frac{1}{3}\pi h^2(3r - h).$$

For a tank of radius 3 metres, find the rate of change of volume when the height is 2 metres and dropping at the rate of 1 centimetre per hour.

5. The volume of a sphere is decreasing at the rate of 6 cm^3/sec. When the radius is 10 cm, how fast is the surface area decreasing?

6. Find the Cartesian equations (equations involving x and y only) corresponding to these parametric equations, and identify the type of curve.

(*i*) $x = 6t - 1, \quad y = 2 - t$

(*ii*) $x = 5t, \quad y = t^2 + t^3$

(*iii*) $x = 2 - \cos\theta, \quad y = 4 - \sin\theta$

(*iv*) $x = 2\cos t, \quad y = 16\sin t$

(*v*) $x = 2 - 6\cos t, \quad y = 1 + 3\sin t$

7. Find parametric equations for the following curves. Give the corresponding values of the parameter.

(*i*) The cubic $y = (x - 1)^3$

(*ii*) The “sideways” parabola $x = y^2$

(*iii*) The circle with centre $(1, 3)$ and radius 2

(*iv*) The ellipse $16(x - 1)^2 + 9(y - 3)^2 = 1$

1.3 Trigonometric Identities

1. (*i*) If $\sin x = \dfrac{7}{25}$ and $\cos x < 0$, $\cos y = \dfrac{9}{41}$ and $\sin y > 0$, find $\sin(x+y)$, $\sin(x-y)$, $\cos(x+y)$, $\cos(x-y)$, $\tan(x+y)$ and $\tan(x-y)$.

(*ii*) If $\cos x = -\dfrac{2\sqrt{5}}{5}$ and $\sin x < 0$, $\sin y = -\dfrac{3\sqrt{10}}{10}$ and $\cos y > 0$, find $\sin(x+y)$, $\sin(x-y)$, $\cos(x+y)$, $\cos(x-y)$, $\tan(x+y)$ and $\tan(x-y)$.

2. Express the following in terms of the argument indicated in parentheses.

(*i*) $\sin 4x$; $(2x)$

(*ii*) $\cos 6x$; $(3x)$

(*iii*) $\tan 10x$; $(5x)$

(*iv*) $1 + \cos 2x$; (x)

(*v*) $1 - \cos 2x$; (x)

(*vi*) $\sin \dfrac{x}{2}$; $\left(\dfrac{x}{4}\right)$

(*vii*) $\cos \dfrac{x}{2}$; $\left(\dfrac{x}{4}\right)$

(*viii*) $(\sin x + \cos x)^2$; $(2x)$

(*ix*) $2\cos^2 3x - 1$; $(6x)$

(*x*) $\cos^2 2x - \sin^2 2x$; $(4x)$

(*xi*) $2 - 2\sin^2 \dfrac{x}{4}$; $\left(\dfrac{x}{2}\right)$

(*xii*) $1 - \cos x$; $\left(\dfrac{x}{2}\right)$

(*xiii*) $(\sin x - \cos x)^2$; $(2x)$

(*xiv*) $2\sin 3x \cos 3x$; $(6x)$

(*xv*) $\sin^2 x$; $(2x)$

(*xvi*) $\cos^2 x$; $(2x)$

3. (*i*) If $\sin x = \dfrac{12}{13}$, find $\sin 2x$ and $\cos 2x$, when $0 < x < \dfrac{\pi}{2}$.

(*ii*) Find the possible values of $\sin\left(\dfrac{x}{2}\right)$ and $\cos\left(\dfrac{x}{2}\right)$ if $\cos x = \dfrac{7}{25}$.

(*iii*) Find $\tan(x+y)$ if $\tan x = \dfrac{1}{2}$ and $\tan y = \dfrac{1}{3}$.
What are the possible values of $(x+y)$, given that $0 < x < 2\pi$ and $0 < y < 2\pi$?
(Hint: use the graph of the tan function to help you here.)

(*iv*) Find $\tan 2x$ if

(a) $\tan x = \dfrac{1}{2}$ (b) $\tan x = 1$

Explain your answer to part (b).

(*v*) What are the possible values of $\tan\left(\dfrac{x}{2}\right)$ if $\tan x = \dfrac{5}{12}$?

(*vi*) Find $\sin 3x$ if $\sin x = \dfrac{1}{4}$.

(*vii*) Find $\cos 3x$ if $\cos x = \dfrac{1}{4}$.

(*viii*) If $0 < x < \frac{\pi}{2}$, $0 < y < \frac{\pi}{2}$, $\tan x = \dfrac{1}{7}$ and $\tan y = \dfrac{1}{3}$, show that $x + 2y = \dfrac{\pi}{4}$.
(Hint: use the graph of the tan function to help you here. You could also more accurately pinpoint the intervals in which x and y must lie.)

4. Using the substitution $t = \tan\left(\frac{x}{2}\right)$, write the following expressions in terms of t.

(i) $\dfrac{2\sin x - \cos x}{\sin^2 x + 1}$

(ii) $\dfrac{3 - \sin x}{2\sin x + \cos x}$

(iii) $\dfrac{1 + \sin x + \cos x}{1 + \tan x}$

(iv) $\dfrac{\sec x}{1 + \sin x}$

5. Find positive r and an appropriate α (in radians) such that each expression is rewritten as a sine function $r\sin(t + \alpha)$.

(i) $\sin t - \cos t$

(ii) $2\sin t + \cos t$

(iii) $2\sin t - 3\cos t$

(iv) $\sin t + \sqrt{2}\cos t$

(v) $a\sin t + b\cos t$

1.4 Mathematical Induction

1. Prove the following proposition:

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

where n is a positive integer.

2. Prove by induction that for all $n \geq 1$,

(i) $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$

(ii) $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

(iii) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(iv) $\sum_{r=1}^{n}(2r-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

3. Prove that $2^n < n!$ for all $n \geq 4$. The notation $n!$ means

$$n! = n(n-1)(n-2)(n-3)\ldots\ldots 3.2.1.$$

4. Prove, by induction, that for all positive integers n,

(i) $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

(ii) $2^n \geq n+1$.

5. Prove that if a and d are any constants, then for all $n \geq 1$,

$$a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) = \frac{n}{2}(2a+(n-1)d)$$

6. Suppose you have n lines in a plane, arranged so that no three of the lines are concurrent, and no two of the lines are parallel. Show that, for $n \geq 1$, n such lines divide the plane into $\frac{1}{2}(n^2+n+2)$ regions.

7. If $f(n) = 5^n - 2^n$, where n is a positive integer, write down the values of $f(1), f(2), f(3), f(4)$. Do all these numbers have a common factor? What general property of the number $5^n - 2^n$ is suggested by these results? Try to prove by induction that the suggested result is true for all positive integers n.

8. Prove that $2^n > n^2$ for all $n \geq 5$.

1.5 Polynomials and rational functions

1. Find the quotient and remainder when:

(*i*) x^3 is divided by $x-2$

(*ii*) x^3-3x^2+7x+2 is divided by $x+1$

(*iii*) x^4+1 is divided by $x+3$

(*iv*) $2x^3+x^2+2$ is divided by $x-4$

(*v*) x^5+2x is divided by $(x-1)(x-2)$

(*vi*) $2x^4-x^3+5$ is divided by $(x+1)(x+2)$

(*vii*) x^6 is divided by $(x-2)^2$

(*viii*) $3x^4+2x-3$ is divided by $(x+1)^3$

2. Find factors of:

(*i*) x^3-7x+6 (*ii*) x^3-x^2-5x+6

(*iii*) $2x^3-3x^2-11x+6$ (*iv*) $6x^3+13x^2-4$

3. For what value of a is $(x+3)$ a factor of $6x^3+ax^2+x-6$?

4. Show that the linear expression is a factor of the given polynomial in each of the following exercises:

(*i*) $x-2$; $3x^4-8x^3+9x^2-17x+14$

(*ii*) $2x-3$; $16x^4+16x^3-64x^2+9$

(*iii*) $x+2$; $6x^3+5x^2-17x-6$

5. If $x+2$ is a factor of $x^3-2x^2+kx-10$, find k and also the remaining factors.

6. Express as a product of two factors:

(*i*) x^5-1 (*ii*) x^5+1 (*iii*) x^5-32

7. Find two intervals of the real numbers, of lengths 0.1, that contain roots of the equation $x^3-x^2-8x+10=0$.

8. Find an interval of the real numbers, of length 0.1, that contains a root of the equation $9x^3-4x^2+9x-4=0$.

9. Draw a graph of each of the following functions.

(*i*) $\dfrac{2}{x-1}$ (*ii*) $\dfrac{1+x}{x}$ (*iii*) $\dfrac{x+1}{x+2}$

(*iv*) $\dfrac{1}{x^2+4}$ (*v*) $\dfrac{1-3x}{x-1}$ (*vi*) $\dfrac{x+3}{x^2-9}$

(*vii*) $\dfrac{x^2}{1-x^2}$ (*viii*) $\dfrac{1}{(x-1)(x-2)(x-3)}$

1.6 Solving equations

1. Solve the following cubic equations by finding a low integer value solution (positive or negative) and then factorising the cubic using long division.

(i)	$2x^3 + 3x^2 - 18x + 8 = 0$	(ii)	$9x^3 - 9x^2 - x + 1 = 0$
(iii)	$x^3 + 3x^2 - 13x - 15 = 0$	(iv)	$x^3 + 12x^2 + 41x + 42 = 0$

2. Solve the following equations given that $-\pi \leq x \leq \pi$.

(i)	$\cos^2 x + \cos x = 0$	(ii)	$2\sin^2 x - \sin x = 0$
(iii)	$\tan^2 x - \tan x = 0$	(iv)	$\sqrt{3} + \tan^2 x = (\sqrt{3} + 1)\tan x$
(v)	$2\sin^2 x - 1 = 0$	(vi)	$2\sin^2 x + 5\cos x - 4 = 0$
(vii)	$\cos^2 x + 5\sin x - 7 = 0.$		

3. Solve $\tan 2\theta = \cot\theta$ when $\theta \in [0, 2\pi]$.

4.

(i)	$1 + 5x > 5 - 3x$	(ii)	$-5 \leq 3 - 2x \leq 9$
(iii)	$x > 1 - x \geq 3 + 2x$	(iv)	$x^2 < 2x + 8$
(v)	$x^2 > 5$	(vi)	$-3 < \frac{1}{x} \leq 1$
(vii)	$\dfrac{2+x}{3-x} \leq 1$	$(viii)$	$\dfrac{x^2 - 2x}{x^2 - 2} > 0$

5. Solve $\dfrac{x^2 - 2x}{2x^2 + 1} > 1$

6. Solve $\dfrac{x^2 - 2x}{2x^2 + 1} \leq 1$

7. Celsius (C) and Fahrenheit (F) temperatures are related by the equation

$$C = \frac{5}{9}(F - 32).$$

(i) What is the range of Celsius temperatures when $50 \leq F \leq 95$?

(ii) What is the range of Fahrenheit temperatures when $20 \leq C \leq 30$?

8. Solve these inequalities.

(i)	$\lvert x - 4\rvert < 0.1$	(ii)	$\lvert 5x - 2\rvert < 6.$
(iii)	$\lvert 2x - 5\rvert \leq \lvert x + 4\rvert$	(iv)	$\dfrac{\lvert 2 - 3x\rvert}{\lvert 1 + 2x\rvert} \leq 4.$

9. Show that if $|x + 3| < \dfrac{1}{2}$ then $|4x + 13| < 3$.

1.7 Integration

1. Use the substitutions indicated to find the indefinite integrals.

(*i*) $\int 2x(x^2-1)^4\,dx; \qquad u = x^2 - 1$

(*ii*) $\int x^2\sqrt{x^3+1}\,dx; \qquad u = x^3 + 1$

(*iii*) $\int \frac{2x}{\sqrt{x^2-4}}\,dx; \qquad u = x^2 - 4$

(*iv*) $\int \frac{dx}{(2x+3)^3}; \qquad u = 2x + 3$

(*v*) $\int x \sin x^2\,dx; \qquad u = x^2$

(*vi*) $\int x e^{x^2-1}\,dx; \qquad u = x^2 - 1$

(*vii*) $\int x(x-1)^3\,dx; \qquad u = x - 1$

(*viii*) $\int x^2(1+x^3)^4\,dx; \qquad u = 1 + x^3$

2. Use the substitutions indicated to evaluate the definite integrals.

(*i*) $\int_0^1 x\sqrt{1-x^2}\,dx; \qquad u = 1 - x^2$

(*ii*) $\int_0^2 \frac{2x}{\sqrt{x^2+1}}\,dx; \qquad u = x^2 + 1$

(*iii*) $\int_3^4 (2x-3)(x^2-3x+2)^2\,dx; \qquad u = x^2 - 3x + 2$

(*iv*) $\int_0^1 x e^{x^2}\,dx; \qquad u = x^2$

(*v*) $\int_0^1 \frac{e^x}{e^x+2}\,dx; \qquad u = e^x + 2$

3. Write down the integral of each the following functions.

(a)	$21x^6$	(b)	$\frac{1}{x^6}$	(c)	$\frac{1}{x}$
(d)	$x^6 + x^3 - \frac{1}{x}$	(e)	$x^{1/2}$	(f)	$\sqrt{x}$
(g)	$(1-3x)^2$	(h)	$(1-x^2)^2$	(i)	$\cos x$
(j)	$\cos 3x$	(k)	$\cos\frac{x}{2}$	(l)	$\sin x$
(m)	$\frac{1}{(5-2x)^4}$	(n)	$\frac{2x}{x^2+3}$	(o)	$2x(x^2+3)$
(p)	$2x\sin(x^2+3)$	(q)	$2xe^{x^2+3}$	(r)	$\frac{x}{x^2+3}$
(s)	$x\sin(x^2+3)$	(t)	$x^2\sin x^3$	(u)	$x^2e^{x^3}$

4. Find the following integrals by the method of integration by substitution.

(i) $\displaystyle\int \sqrt{x}\cos(x\sqrt{x})dx$ (ii) $\displaystyle\int e^x\cos(e^x)dx,$

(iii) $\displaystyle\int x\sqrt{x-4}dx.$ (iv) $\displaystyle\int \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$

(v) $\displaystyle\int \frac{dx}{\sqrt{2x+5}}$ (vi) $\displaystyle\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^3}$

5. Person A states that $\displaystyle\int_{-2}^{1} 2x^2\,dx$ is obviously positive. Person B says that it's negative, since if we put $u = x^2$ then $du = 2x\,dx$ and so

$$\int_{-2}^{1} 2x^2\,dx = \int_{-2}^{1} x.2x\,dx = \int_{4}^{1} \sqrt{u}\,du = -\int_{1}^{4} \sqrt{u}\,du,$$

and this is obviously negative. Who is right?

6. Find the following indefinite integrals.

(i) $\displaystyle\int 2\cos^2 x\,dx$ (ii) $\displaystyle\int \sin^2\frac{x}{2}\,dx$

(iii) $\displaystyle\int \sin^2 x\cos x\,dx$ (iv) $\displaystyle\int \sin^4 x\,dx$

7. Evaluate the following definite integrals.

(i) $\displaystyle\int_0^{\pi/4} \sin^2 x\,dx$ (ii) $\displaystyle\int_{\pi/6}^{\pi/3} \cos^2 x\,dx$

1.8 Inverse Functions

1. Find a largest possible domain over which the following functions are strictly increasing.

(i) x^2-4 (ii) $\sqrt{x^2-4}$ (iii) $\sqrt{4-x^2}$

(iv) $3x-x^2$ (v) x^2+6x+8 (vi) $\dfrac{-1}{x+2}$

2. Show that the following pairs of functions are inverses, by showing that $f(g(x)) = g(f(x)) = x$.

(i) $f(x) = 3x+1;\quad g(x) = \dfrac{x-1}{3}$

(ii) $f(x) = \dfrac{e^x}{2};\quad g(x) = \ln(2x),\ x>0$

(iii) $f(x) = \cos 2x,\quad 0 \le x \le \pi/2;\quad g(x) = \dfrac{\cos^{-1} x}{2},\quad -1 \le x \le 1$

(iv) $f(x) = x^2+1,\quad x \ge 0;\quad g(x) = \sqrt{x-1},\ x \ge 1$

3. Find the inverse function of each of the following functions, and specify the domain of the inverse. Sketch each function and its inverse.

(i) $f(x) = 2x-4,\ x \in \mathbb{R}$ (ii) $f(x) = x^2-1,\quad x \ge 0$

(iii) $f(x) = e^{x/2},\ x \in \mathbb{R}$ (iv) $f(x) = \ln(x-2),\quad x \ge 3$

(v) $f(x) = (x+2)^2,\quad x \le -2$ (vi) $f(x) = x^3+8,\quad -1 \le x \le 1$

(vii) $f(x) = \sin 3x,\quad -\dfrac{\pi}{6} \le x \le \dfrac{\pi}{6}$

4. Find a formula for the inverse function. Restrict the domain if necessary.

(i) $f(x) = x^2-4x+7$ (ii) $f(x) = \ln(x+2) - \ln(x+1)$

5. Find the derivatives of the following functions:

(i) $\sin^{-1}\sqrt{x}$ (ii) $\sin^{-1}(1+x)$ (iii) $\left(\sin^{-1} x\right)^2$

(iv) $\sin^{-1}(x^2)$ (v) $\tan^{-1}\sqrt{x}$ (vi) $\tan^{-1}(\ln x)$

6. Find the following integrals:

(i) $\displaystyle\int \frac{1}{1+x^2}\,dx$ (ii) $\displaystyle\int \frac{\sin x}{1+\cos^2 x}\,dx$ (iii) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

(iv) $\displaystyle\int \frac{dx}{13-4x+x^2}$ (v) $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}}$

1.9 Applications

1. Show that $x = 1 + 2e^{-3t}$ satisfies the differential equation $\dfrac{dx}{dt} = -3(x-1)$.

2. Show that $y = 5 - e^{-x}$ satisfies the differential equation $\dfrac{dy}{dx} = -(y-5)$.

3. Suppose that a murdered body cools from 37°C to 34°C in two hours, in a room where the temperature is a constant 18°C. Assume Newton's law of cooling.

- *(i)* Find the temperature of the body as a function of time.
- *(ii)* Sketch a graph of temperature against time.
- *(iii)* What happens to the temperature of the body in the long run?
- *(iv)* Suppose the body is found at 4pm, and the temperature is then 30°C. At what time was the murder committed?

4. A potato, at 23°C, is put into a 200°C oven. Assume Newton's law of heating.

- *(i)* Find an equation for the temperature of the potato as a function of time, assuming that after 30 minutes its temperature is 120°C.
- *(ii)* Draw a graph of temperature of the potato against time.

5. A metal bar has a temperature of 1230°C and cools to 1030°C in 10 minutes when the surrounding temperature is 30°C. Assume Newton's law of cooling. How long will it take to cool to 80°C?

6. When a cold drink is taken from a refrigerator its temperature is 4°C. After 25 minutes in a 20°C room its temperature has increased to 10°C. Assume Newton's law of heating.

- *(i)* What is the temperature of the drink after 50 minutes?
- *(ii)* When will its temperature be 19.5°C?

7. Show that the functions $x = 5\sin 2t$, $x = 8\cos 2t$, $x = 3\sin(2t + \pi)$ and $x = \cos(2t - \pi/2)$ are all solutions to the differential equation $\dfrac{d^2x}{dt^2} = -4x$.

8. The equation of motion of a particle is $\dfrac{d^2x}{dt^2} = -9x$. Find its period, amplitude and maximum speed given that $x = 0$ and $\dfrac{dx}{dt} = 2$ when $t = 0$.

9. A particle moves in a straight line and its position at time t is given by $x = 5\sin\left(\frac{\pi}{2}t + \frac{\pi}{6}\right)$.

- *(i)* Show that the particle is undergoing simple harmonic motion.
- *(ii)* Find the period and amplitude of the motion.
- *(iii)* Find the acceleration of the particle when $x = -4$.
- *(iv)* After time $t = 0$, when does the particle first reach maximum velocity?

10. A particle moves in a straight line and its position at time t is given by $x = 8\sin 2t + 6\cos 2t$.

(i) Show that the particle is undergoing simple harmonic motion.

(ii) Find the period and amplitude of the motion.

1.10 Counting and Permutations

1. How many ways are there from one vertex of a cube to the opposite vertex, each way being along three edges of the cube?

2. How many four-figure whole numbers may be formed from the digits $1, 2, 3, 4$? How many of these have no repeated digits?

3. In how many ways is it possible to fill the four top places in a league of ten sports teams?

4. In how many ways is it possible to letter the vertices of a hexagon ABCDEF if consecutive vertices are lettered alphabetically?

5. How many different outfits (i.e. one garment to cover the top, and one garment to cover the bottom) can be created from 6 different T-shirts, 3 different pairs of shorts and 4 different skirts?

6. Express the following as quotients of factorials:

$$7 \times 6 \times 5, \qquad 8 \times 9 \times 10 \times 11, \qquad \frac{1}{6 \times 5 \times 4},$$

$$\frac{13 \times 12 \times 11}{1 \times 2 \times 3}, \qquad \frac{n(n-1)(n-2)}{1 \times 2 \times 3}.$$

7. Simplify:

$$\frac{r!}{(r-1)!}, \qquad \frac{r! + (r+1)!}{(r-1)!}, \qquad r! - (r-1)!.$$

8. In how many ways may ten boys be seated on a bench so that two particular boys are always seated together?

9. Evaluate

$$^{7}P_5, \qquad ^{8}P_3, \qquad ^{10}P_2, \qquad \frac{^{6}P_4}{^{3}P_3}.$$

10. Express in terms of n or in terms of n and r:

$$^{n+1}P_4, \qquad ^{2n}P_3, \qquad ^{n}P_{r-1}, \qquad \frac{^{n+1}P_r}{^{n}P_{r-1}}.$$

11. Find n if

(*i*) $^{n-1}P_2 = 56$

(*ii*) $^{n}P_4 = 90 \times ^{n-2} P_2$

(*iii*) $^{n+1}P_5 = 72 \times ^{n-1} P_3$

12. How many whole numbers of four digits can be made with the numerals 1, 2, 4, 7, 8, 9, if each digit is used
(i) any number of times,
(ii) not more than once, in any given number?

13. How many whole numbers greater than 5 000 and without repeated digits can be formed with the numerals $2, 3, 5, 8, 9, 0$? How many of these are multiples of ten?

14. How many whole numbers of five different digits and greater than 10 000 can be formed with the numerals $0, 1, 2, 3, 4, 5$? Of these how many
(i) are even,
(ii) lie between 30 000 and 50 000?

15. A branch line on a railway network has 10 stations. How many single tickets must be printed for this line?

16. In how many ways can six people sit together in a row if
(i) two of them must always be together,
(ii) three of them must always be together?

17. Using each digit not more than once in any number, how many numbers between 3 000 and 6 000 can be formed with the digits $2, 3, 5, 7, 8, 9$? How many of these numbers are even?

18. How many ways are there of displaying
(i) five different flags on one mast,
(ii) four different flags on two masts?

19. A car number plate consists of a block of letters followed by a block of numerals. If the letters span the alphabet and the numerals are from the digits $0, 1, 2, \ldots, 9$, how many such index numbers are there containing three letters and three numbers?

1.11 Combinations and the binomial theorem

1. Evaluate ${}^6C_4$, ${}^{20}C_{18}$, ${}^{15}C_{10}$.

2. (i) Find a positive integer n such that ${}^nC_{15} = {}^n C_8$.
(ii) If ${}^nC_3 = 8 \times {}^n C_2$, find the positive integer n.

3. A school is divided into 8 houses for sporting competitions. Each house plays each other house once. How many matches are played?

4. Three prefects are to be chosen from 10 prefects for daily duty. How many days is it before a particular group of prefects must serve again?

5. How many committees of six may be chosen from 10 persons A,B,C,..., if each committee
(i) includes A,
(ii) excludes B,
(iii) includes A *and* excludes B,
(iv) includes A or B (or both)?

6. In how many ways can a group of ten men be divided into one group of 3 and and one group of 7?

7. In how many ways can 12 different objects be divided into
(i) one group of 8 and one group of 4,
(ii) a group of 7, a group of 3, and a group of 2?

8. If S is the set $\{a_1, a_2, a_3, a_4\}$, how many proper subsets of S are there? (A proper subset is one with fewer than 4 elements. One of these subsets contains no elements at all; it is called the *empty* set.)

9. If A is the set $\{a, b, c, d, e\}$, write down the complementary subset of each of the following subsets of A: $\{a, b\}$, and $\{b, d, e\}$. How many subsets of A are there with (a) 3 elements, (b) 2 elements?

10. Verify that $\binom{23}{0} + \binom{23}{1} + \binom{23}{2} + \binom{23}{3} = 2^{11}$.

11. Write down the expansions of:

(i) $(1-x)^5$	*(ii)* $(2+x)^4$	*(iii)* $(3-2x)^3$
(iv) $(a-b)^4$	*(v)* $\left(x+\frac{1}{x}\right)^5$	*(vi)* $\left(x-\frac{1}{x}\right)^5$
(vii) $(x^2+3y)^3$	*(viii)* $(2+x^3)^4$	

12. In the expansion of each of the following, find the coefficient of the specified power of x:

(i) $(1+2x)^7$; x^4	*(ii)* $(3x-2x^3)^5$; x^{11}	*(iii)* $(5x-3)^7$; x^4
(iv) $\left(x-\frac{2}{x}\right)^8$; x^2	*(v)* $\left(x^2+\frac{4}{x}\right)^{10}$; x^{-1}	

13. Show that each of the following is a rational number, and find that number:

(*i*) $(\sqrt{2}+1)^5 - (\sqrt{2}-1)^5$

(*ii*) $(\sqrt{3}+1)^6 + (\sqrt{3}-1)^6$

(*iii*) $(\sqrt{3}+\sqrt{2})^4 + (\sqrt{3}-\sqrt{2})^4$

14. Find the coefficient of:

(*i*) x^6 in the expansion of $(2+x)(1+x)^8$

(*ii*) x^5 in the expansion of $(1+x)(2-x)^7$

(*iii*) x^8 in the expansion of $(2-x^2)(3+x)^9$

(*iv*) x^4 in the expansion of $(1-x+x^2)(2+x)^6$

15. The third term of the expansion of $(1+x)^n$ is $36x^2$. Find the second term (n is a positive integer).

16. Show that the coefficient of x^2 in the expansion of $(2+x)(1+x)^n$ is n^2 and the coefficient of x^3 is $\dfrac{1}{6}n(n-1)(2n-1)$.

17. Find the coefficient of x^r in the expansion of $(2-x)(1+x)^8$ and hence show that one of the coefficients of the expansion is zero.

18. Find the constants k and n if the expansion of $(1+kx)^n$ begins

$$1 - 12x + 60x^2 - \cdots .$$

19. Use the binomial theorem to find the value of $(3.01)^6$ correct to the first decimal place.

20. (Harder) Two successive numbers in the n^{th} row of Pascal's triangle are 455 and 1 365. Find n.

Chapter 2

Answers to exercises

2.1 Functions I

1. (*i*) $\{x \mid x \neq -1\}$ in set notation, or $(-\infty,-1)\cup(-1,\infty)$ in interval notation.

(*ii*) $(-\infty,-2)\cup(-2,0)\cup(0,\infty)$.

(*iii*) $\mathbb{R}$

(*iv*) $[3.5,\infty)$, that is, $3.5 \le x < \infty$. The range is $[0,\infty)$.

(*v*) $\mathbb{R}$. The range is also $\mathbb{R}$.

(*vi*) $[-2,-\sqrt{3}]\cup[\sqrt{3},2]$

2. (*i*) $f(g(x)) = -12x-16$, domain $\mathbb{R}$, $g(f(x)) = -12x-14$, domain $\mathbb{R}$

(*ii*) $f(g(x)) = \dfrac{1}{\sqrt{x^2-1}}$, domain $(1,\infty)\cup(-\infty,-1)$, $g(f(x)) = \dfrac{\sqrt{1-x^2}}{|x|}$, domain $(0,1]\cup[-1,0)$

(*iii*) $f(g(x)) = \sqrt{x^2-1}$, domain $|x|\ge 1$, $g(f(x)) = x-1$, domain $[1,\infty)$

(*iv*) $f(g(x)) = \dfrac{x}{x+2}$, domain $[-1,\infty)$, $g(f(x)) = \dfrac{|x|\sqrt{2}}{\sqrt{x^2+1}}$, domain $\mathbb{R}$

(*v*) $f(g(x)) = x^2-1$, domain $\mathbb{R}$, $g(f(x)) = |x^2-1|$, domain $\mathbb{R}$

3. (*i*)

(a) $2x$ (b) $12-2x$ (c) $-2x^{-3}$

(d) $-11(2x-3)^{-2}$ (e) $\dfrac{1}{2\sqrt{x}}$ (f) $-\dfrac{1}{2x^{3/2}}$

(g) $1+\dfrac{1}{x^2}$ (h) $-\dfrac{3}{x^4}+\dfrac{1}{2x^{1/2}}$

(*ii*)

(a) $6x^5-16x^3+6x$ (b) $x(2\sin x + x\cos x)$

(c) $x(2\tan x + x\sec^2 x)$ (d) $e^x(x+4)$

(e) $-\dfrac{x+2}{e^x}$ (f) $\dfrac{\cos x}{x}-\dfrac{\sin x}{x^2}$

(g) $x^2e^x(3\sin x + x(\sin x+\cos x))$ (h) $\dfrac{2\sec^2 x}{(1-\tan x)^2}$

(i) $\dfrac{1+\sin x}{\cos^2 x} = \dfrac{1}{1-\sin x}$ (j) $\dfrac{2+\sqrt{x}}{2(1+\sqrt{x})^2}$

4. (*i*) $dy/dx = -3x^2 + 6x + 24$.

(*ii*) Gradient is $8x - 27/x^2$, $x = 3/2$, minimum.

(*iii*) Tangent $y = 8x - 13$; normal $8y = -x + 91$.

(*iv*) The tangents have slopes 2 and $-1/2$.

(*v*) $V = 9.2$ knots.

(*vi*) $\alpha = \sqrt{b/a}$.

2.2 Functions II

1. (*i*) $\dfrac{dy}{dx} = -\cos x\,\sin(\sin x)$ (*ii*) $\dfrac{dy}{dx} = -\sin x\,\cos(\cos x)$

(*iii*) $\dfrac{dy}{dx} = 3\cos x\,\sin x\,(\sin x + \cos x)$ (*iv*) $\dfrac{dy}{dx} = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$

(*v*) $\dfrac{dy}{dx} = \dfrac{2 + \frac{x}{2}}{(2+x)^{3/2}}$ (*vi*) $\dfrac{dy}{dx} = \dfrac{1}{2}\left(1 + \dfrac{1}{2\sqrt{x}}\right)\dfrac{1}{\sqrt{x + \sqrt{x}}}$

(*vii*) $\dfrac{dy}{dx} = 2x\sec^2 x^2 + 2\tan x\sec^2 x$

2. (*i*) $\dfrac{dy}{dx} = \dfrac{2y - x^2}{y^2 - 2x}$; at $(3,3)$ this equals -1.

(*ii*) $\dfrac{dy}{dx} = \dfrac{3x^2 - 2x^3}{y}$; at $(1,1)$ this equals 1.

(*iii*) $\dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{\frac{1}{3}}$; at $(-3\sqrt{3}, 1)$ this equals $\dfrac{1}{\sqrt{3}}$.

(*iv*) $\dfrac{dy}{dx} = -\dfrac{y}{x}$; at $(1,1)$ this equals -1.

3. Area increases at the rate of 1 cm^2/sec. Perimeter increases at the rate of $\dfrac{4}{5\sqrt{5}}$ cm/sec when $h = 10$ cm.

4. Volume is decreasing at the rate of $\dfrac{8\pi}{100}$ cubic metres per hour.

5. Surface area is decreasing at the rate of 1.2 cm^2/sec.

6. (*i*) Line $y = \frac{11}{6} - \frac{1}{6}x$

(*ii*) Cubic curve $y = \dfrac{x^2}{25} + \dfrac{x^3}{125}$

(*iii*) Circle of radius 1 centred at $(2,4)$, with equation

$$(2-x)^2 + (4-y)^2 = 1.$$

(*iv*) Ellipse centred at the origin, x intercepts $(\pm 2, 0)$, y intercepts $(0, \pm 16)$, with equation

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{16}\right)^2 = 1.$$

(*v*) Ellipse centred at $(2,1)$,

$$\left(\frac{x-2}{-6}\right)^2+\left(\frac{y-1}{3}\right)^2=1.$$

7. There are many ways to parametrise a given curve. Here are some.

(*i*) $x=t,\ y=(t-1)^3, \quad t\in\mathbb{R}.$

(*ii*) $y=t,\ x=t^2, \quad t\in\mathbb{R}.$

(*iii*) $x=1+2\cos t,\ y=3+2\sin t, \quad t\in[0,2\pi).$

(*iv*) $x=1+\frac{1}{4}\cos t,\ y=3+\frac{1}{3}\sin t, \quad t\in[0,2\pi).$

2.3 Trigonometric identities

1. (*i*) $-\dfrac{897}{1025}, \quad \dfrac{1023}{1025}, \quad -\dfrac{496}{1025}, \quad \dfrac{64}{1025}, \quad \dfrac{897}{496}, \quad \dfrac{1023}{64}.$

(*ii*) $\dfrac{1}{\sqrt{2}}, \quad -\dfrac{7}{5\sqrt{2}}, \quad -\dfrac{1}{\sqrt{2}}, \quad \dfrac{1}{5\sqrt{2}}, \quad -1, \quad -7.$

2. (*i*) $2\sin 2x\cos 2x$

(*ii*) $\cos^2 3x-\sin^2 3x$, etc.

(*iii*) $\dfrac{2\tan 5x}{1-\tan^2 5x}$

(*iv*) $2\cos^2 x$

(*v*) $2\sin^2 x$

(*vi*) $2\sin\dfrac{x}{4}\cos\dfrac{x}{4}$

(*vii*) $\cos^2\dfrac{x}{4}-\sin^2\dfrac{x}{4}$

(*viii*) $1+\sin 2x$

(*ix*) $\cos 6x$

(*x*) $\cos 4x$

(*xi*) $1+\cos\dfrac{x}{2}$

(*xii*) $2\sin^2\dfrac{x}{2}$

(*xiii*) $1-\sin 2x$

(*xiv*) $\sin 6x$

(*xv*) $\dfrac{1-\cos 2x}{2}$

(*xvi*) $\dfrac{1+\cos 2x}{2}$

3. (*i*) $\dfrac{120}{169}, -\dfrac{119}{169}$

(*ii*) $\dfrac{3}{5}$ or $-\dfrac{3}{5}$, $\dfrac{4}{5}$ or $-\dfrac{4}{5}$

(*iii*) $1;\ \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}$

(*iv*) (a) $\dfrac{4}{3}$ (b) undefined.

(*v*) $\dfrac{1}{5}$ or -5

(*vi*) $\dfrac{11}{16}$

(*vii*) $-\dfrac{11}{16}$

(*viii*) $\tan(x+2y)=1$ and $0<x+2y<\dfrac{3\pi}{4}$

4. (*i*) $\dfrac{t^4+4t^3+4t-1}{t^4+6t^2+1}$

(*ii*) $\dfrac{3t^2-2t+3}{-t^2+4t+1}$

(*iii*) $\dfrac{-2t^3-2t^2+2t+2}{-t^4+2t^3+2t+1}$

(*iv*) $\dfrac{(1+t^2)^2}{1+2t-2t^3-t^4}$

5. (*i*) $r = \sqrt{2},\ \alpha = -\pi/4$ (*ii*) $r = \sqrt{5},\ \alpha \approx 0.464$
(*iii*) $r = \sqrt{13},\ \alpha \approx -0.983$ (*iv*) $r = \sqrt{3},\ \alpha \approx 0.955$
(*v*) $r = \sqrt{a^2 + b^2},\ \tan\alpha = b/a$

2.4 Mathematical induction

1. When $n = 1$, LHS$= 1$, RHS$= 1^2 = 1$ and so the proposition is true when $n = 1$. Now we assume that

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

and use this to prove that

$$1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = (n + 1)^2.$$

Now

$$1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = n^2 + 2n + 1 = (n + 1)^2$$

and this is the result we want. We conclude that the proposition is true for all positive integers.

3. When $n = 4$, LHS$= 2^4 = 16$, RHS$= 4! = 24$ and it is certainly true that $16 < 24$. Thus the proposition is true for $n = 4$. Now we assume that $2^n < n!$ and use this to prove that $2^{n+1} < (n + 1)!$. We have

$$2^{n+1} = 2.2^n < 2.n! < (n + 1)n! = (n + 1)!$$

Hence the proposition is true for all integers $n \geq 4$.

5. When $n = 1$, LHS$= a$, RHS$= \frac{1}{2}(2a) = a$ and the proposition is true. Now we assume that

$$a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d) = \frac{n}{2}(2a + (n - 1)d)$$

We must use this to prove that

$$a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d) + (a + nd) = \frac{n + 1}{2}(2a + nd).$$

Now

$$\begin{aligned}
&a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d) + (a + nd)\\
&= \frac{n}{2}(2a + (n - 1)d) + (a + nd)\\
&= na + \frac{n(n - 1)d}{2} + a + nd\\
&= (n + 1)a + \frac{d}{2}(n(n - 1) + 2n)\\
&= (n + 1)a + \frac{d}{2}(n(n + 1))\\
&= (n + 1)(a + \frac{nd}{2})\\
&= \frac{n + 1}{2}(2a + nd)
\end{aligned}$$

Therefore the proposition is true for all $n \geq 1$.

7. $f(1) = 3,\ f(2) = 21,\ f(3) = 117,\ f(4) = 609$. All these numbers are multiples of 3. We conjecture the proposition that $f(n)$ is a multiple of 3 for all $n \geq 1$. We have already proved this is true for $n = 1$. We now assume that $f(n)$ is a multiple of 3 and use this to prove that $f(n+1)$ is also a multiple of 3.

$$f(n+1) = 5^{n+1} - 2^{n+1} = 5.5^n - 2.2^n.$$

Now by assumption, $f(n) = 5^n - 2^n = 3k$, for some integer k. Therefore

$$\begin{aligned} f(n+1) &= 5(3k + 2^n) - 2.2^n \\ &= 15k + 2^n(5-2) \\ &= 15k + 3.2^n \\ &= 3(5k + 2^n) \end{aligned}$$

We have now proved that $f(n+1)$ is a multiple of 3, and therefore the proposition is true.

2.5 Polynomials and rational functions

1. (*i*) The quotient is $x^2 + 2x + 4$; the remainder is 8.

(*ii*) $x^2 - 4x + 11; \quad -9$

(*iii*) $x^3 - 3x^2 + 9x - 27; \quad 82$

(*iv*) $2x^2 + 9x + 36; \quad 146$

(*v*) $x^3 + 3x^2 + 7x + 15; \quad 33x - 30$

(*vi*) $2x^2 - 7x + 17; \quad -37x - 29$

(*vii*) $x^4 + 4x^3 + 12x^2 + 32x + 80; \quad 192x - 320$

(*viii*) $3x - 9; \quad 18x^2 + 26x + 6$

2. (*i*) $(x-1)(x-2)(x+3)$

(*ii*) By trial and error (using low integer values) we see that when $x = 2$, $x^3 - x^2 - 5x + 6 = 0$. Therefore $x - 2$ is a factor of $x^3 - x^2 - 5x + 6$. This means that $x^3 - x^2 - 5x + 6 = (x-2)(x^2 + ax - 3)$ where a can be found by expanding the right hand side. We get

$$x^3 - x^2 - 5x + 6 = x^3 + (a-2)x^2 - (3+2a)x + 6$$

which gives $a = 1$. The required factorisation is therefore

$$(x-2)\left(x + \frac{1+\sqrt{13}}{2}\right)\left(x + \frac{1-\sqrt{13}}{2}\right)$$

(*iii*) $(2x-1)(x+2)(x-3)$

(*iv*) $(x+2)(3x+2)(2x-1)$

3. Let $f(x) = 6x^3 + ax^2 + x - 6$. By the remainder theorem, we require $f(-3) = 0$ for $x + 3$ to be a factor of $f(x)$. Now $f(-3) = 9a - 171 = 0$ if and only if $a = 19$.

4. (*i*) Substituting $x = 2$ into $3x^4 - 8x^3 + 9x^2 - 17x + 14$ gives 0, and so $x - 2$ is a factor of the quartic polynomial. The other parts are similar.

5. $k = -13$; $(x-5)(x+1)(x+2)$

6. (*i*) $(x^4 + x^3 + x^2 + x + 1)(x - 1)$

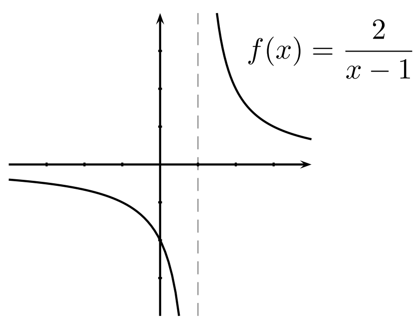
(*ii*) $(x^4 - x^3 + x^2 - x + 1)(x + 1)$

(*iii*) $(x^4 + 2x^3 + 4x^2 + 8x + 16)(x - 2)$

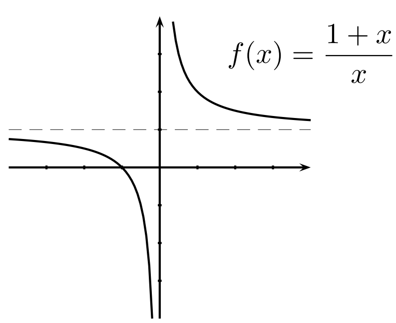
7. (2.5, 2.6), (−3, −2.9). There is also a solution near $x = 1.31$.

8. (0.4, 0.5)

9. (*i*)



(*ii*)



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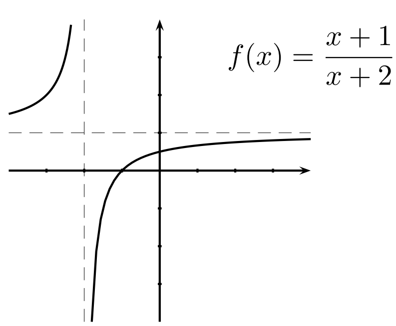

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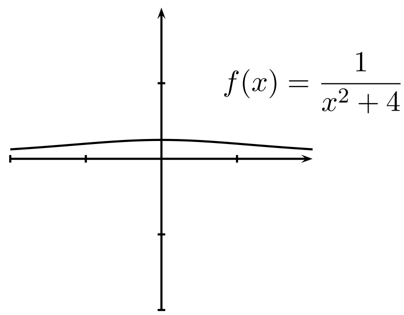

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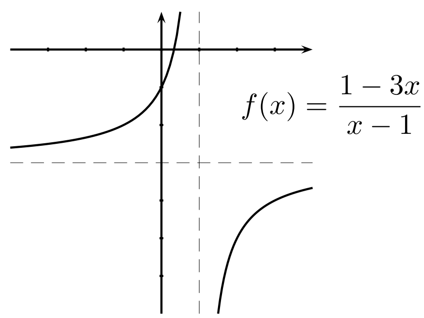

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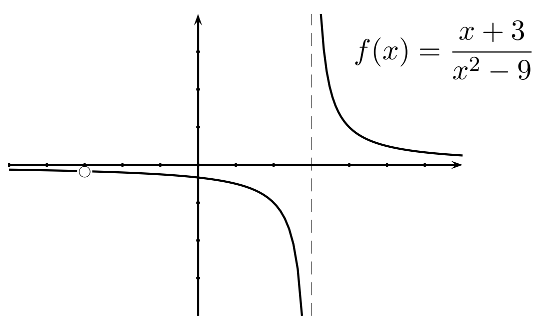

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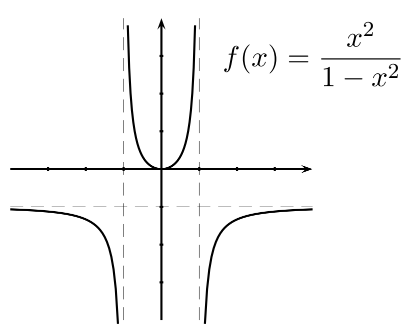
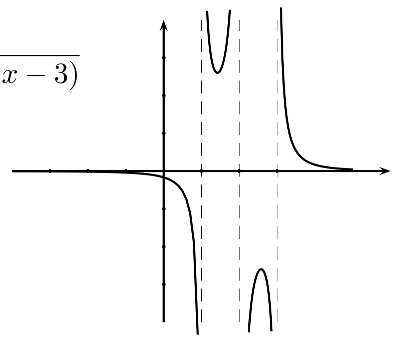


(viii)

$$f(x) = \frac{1}{(x-1)(x-2)(x-3)}$$



2.6 Solving equations

1. (i) Equation factorises as $(x-2)(2x^2+7x-4) = 0$. The solutions are $x = 2,\ -4,\ \frac{1}{2}$.

(ii) Equation factorises as $(x-1)(9x^2-1) = 0$. The solutions are $x = 1, \frac{1}{3} - \frac{1}{3}$.

(iii) Equation factorises as $(x+1)(x^2+2x-15) = 0$. The solutions are $x = -1,\ -5,\ 3$.

(iv) Equation factorises as $(x+2)(x^2+10x+21) = 0$. The solutions are $x = -2,\ -3,\ -7$.

2. (i) $-\pi,\ -\frac{\pi}{2},\ \frac{\pi}{2},\ \pi$. (ii) $-\pi,\ 0,\ \frac{\pi}{6},\ \frac{5\pi}{6},\ \pi$.

(iii) $-\pi,\ 0,\ \pi,\ -\frac{3\pi}{4},\ \frac{\pi}{4}$ (iv) $-\frac{3\pi}{4},\ \frac{\pi}{4},\ \frac{\pi}{3},\ -\frac{2\pi}{3}$.

(v) $-\frac{3\pi}{4},\ -\frac{\pi}{4},\ \frac{\pi}{4},\ \frac{3\pi}{4}$. (vi) $-\frac{\pi}{3},\ \frac{\pi}{3}$.

(vii) No solutions.

3. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

4. Answers are given using interval notation.

(i) $(\frac{1}{2}, \infty)$ (ii) $[-3, 4]$

(iii) No solution (iv) $(-2, 4)$

(v) $(-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)$ (vi) $(-\infty, -\frac{1}{3}) \cup [1, \infty)$

(vii) $(-\infty, \frac{1}{2}] \cup (3, \infty)$ (viii) $(-\infty, -\sqrt{2}) \cup (0, \sqrt{2}) \cup (2, \infty)$

5. No solution

6. All $x \in \mathbb{R}$.

7. (i) $10 \leq C \leq 35$ (ii) $68 \leq F \leq 86$

8. (i) $(3.9, 4.1)$ (ii) $(-0.8, 1.6)$ (iii) $[\frac{1}{3}, 9]$ (iv) $(-\infty, -\frac{6}{5}] \cup [-\frac{2}{11}, \infty)$

9. We are given that $-\frac{1}{2} < x + 3 < \frac{1}{2}$. Multiplying through by 4 then adding 1, we get

$$-2 < 4x + 12 < 2, \quad -1 < 4x + 13 < 3.$$

Therefore $|4x + 13| < 3$.

2.7 Integration techniques

1. (*i*) $\frac{1}{5}\left(x^2-1\right)^5+C$ (*ii*) $\frac{2}{9}\left(x^3+1\right)^{3/2}+C$

(*iii*) $2\sqrt{x^2-4}+C$ (*iv*) $-\frac{1}{4}\frac{1}{(2x+3)^2}+C$

(*v*) $-\frac{1}{2}\cos x^2+C$ (*vi*) $\frac{1}{2}e^{x^2-1}+C$

(*vii*) $\frac{1}{5}(x-1)^5+\frac{1}{4}(x-1)^4+C$ (*viii*) $\frac{1}{15}(1+x^3)^5+C$

2. (*i*) $\frac{1}{3}$ (*ii*) $2\sqrt{5}-2$ (*iii*) $\frac{208}{3}$

(*iv*) $\frac{1}{2}e-\frac{1}{2}$ (*v*) $\ln\left|\frac{e+2}{3}\right|$

3. (a) $3x^7+C$ (b) $-\frac{1}{5x^5}+C$

(c) $\ln|x|+C$ (d) $\frac{x^7}{7}+\frac{x^4}{4}-\ln|x|+C$

(e) $\frac{2x^{\frac{3}{2}}}{3}+C$ (f) $\frac{2x^{\frac{3}{2}}}{3}+C$

(g) $-\frac{(1-3x)^3}{9}+C$ (h) $x-\frac{2x^3}{3}+\frac{x^5}{5}+C$

(i) $\sin x+C$ (j) $\frac{\sin 3x}{3}+C$

(k) $2\sin\frac{x}{2}+C,$ (l) $-\cos x+C$

(m) $\frac{1}{6(5-2x)^3}+C$ (n) $\ln(x^2+3)+C$

(o) $\frac{(x^2+3)^2}{2}+C$ (p) $-\cos(x^2+3)+C$

(q) $e^{x^2+3}+C$ (r) $\frac{1}{2}\ln(x^2+3)+C$

(s) $-\frac{1}{2}\cos(x^2+3)+C$ (t) $-\frac{\cos x^3}{3}+C$

(u) $\frac{e^{x^3}}{3}+C$

4. (*i*) $\frac{2}{3}\sin(x\sqrt{x})+C$ (*ii*) $\sin(e^x)+C$

(*iii*) $\frac{2}{5}(x-4)^{\frac{5}{2}}+\frac{8}{3}(x-4)^{\frac{3}{2}}+C$ (*iv*) $2e^{\sqrt{t}}+C$

(*v*) $\sqrt{2x+5}+C$ (*vi*) $-\frac{1}{(1+\sqrt{x})^2}+C$

5. Person A is correct. What is going wrong with the substitution $u=x^2$ is that it is not correct to say that $x=\sqrt{u}$ (a non-negative quantity) over the whole interval $[-2,1]$, since x is negative on $[-2,0)$.

6. (i) $x+\dfrac{1}{2}\sin 2x+C$ (ii) $\dfrac{1}{2}x-\dfrac{1}{2}\sin x+C$

(iii) $\dfrac{1}{3}\sin^3 x+C$ (iv) $\dfrac{3}{8}x+\dfrac{1}{32}\sin 4x-\dfrac{1}{4}\sin 2x+C$

7. (i) $\dfrac{\pi}{8}-\dfrac{1}{4}$ (ii) $\dfrac{\pi}{12}$

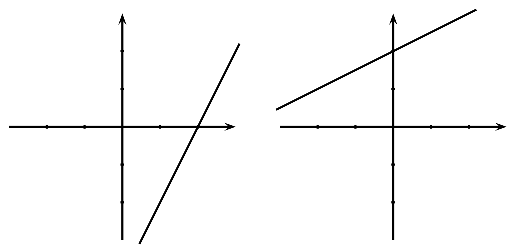
2.8 Inverse functions

1. (i) $x \geq 0$ (ii) $x \geq 2$ (iii) $-2 \leq x \leq 0$

(iv) $x \leq \frac{3}{2}$ (v) $x \geq -3$ (vi) $x \neq -2$

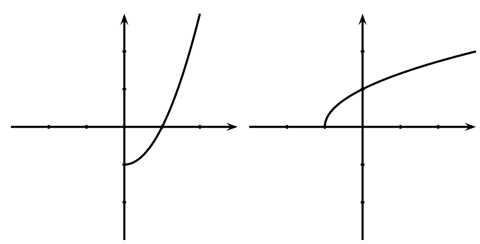
3. (i)

$f^{-1}(x)=\dfrac{x+4}{2};\ \mathbb{R}$



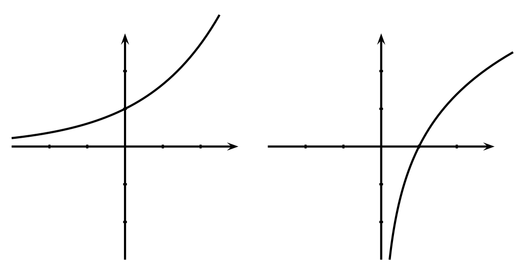
(ii)

$f^{-1}(x)=\sqrt{x+1};\ \ x \geq -1$



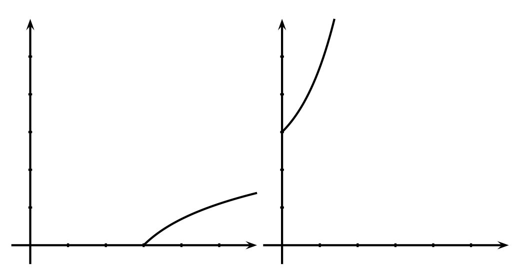
(iii)

$f^{-1}(x)=2\ln x;\ \ x>0$

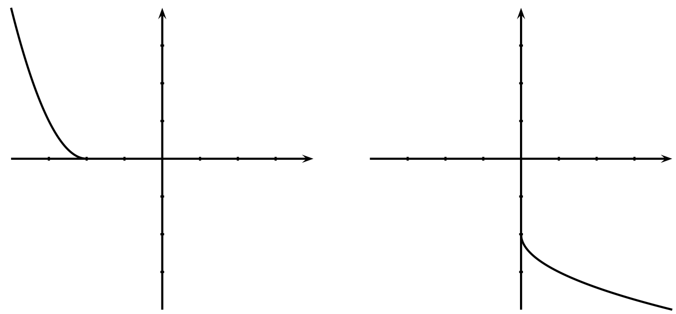


(iv)

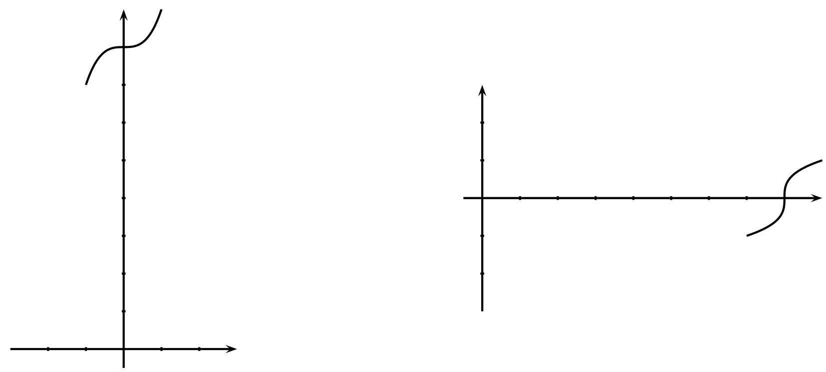
$f^{-1}(x)=e^x+2;\ \ x \geq 0$



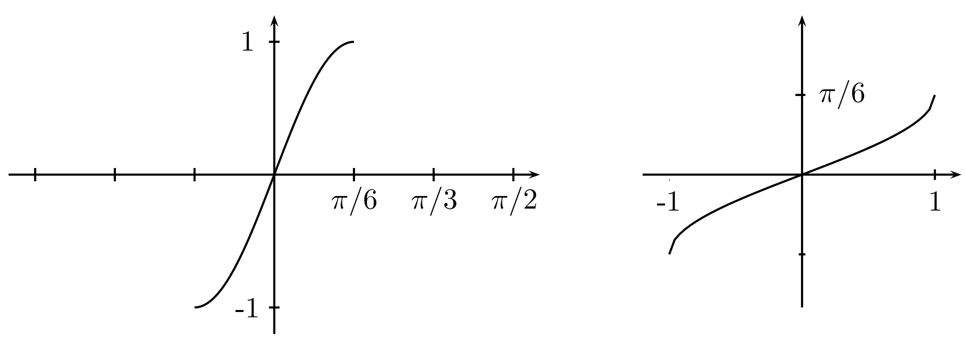
$(v)\quad f^{-1}(x) = -\sqrt{x} - 2;\;\; x \geq 0$



$(vi)\quad f^{-1}(x) = (x-8)^{\frac{1}{3}};\;\; 7 \leq x \leq 9$



$(vii)\quad f^{-1}(x) = \dfrac{\sin^{-1} x}{3};\;\; -1 \leq x \leq 1$



4. (i) Restricting the domain to $[2, \infty)$, the inverse function is $f^{-1}(x) = 2 + \sqrt{x-3}$, $x \geq 3$.

(ii) The function has an inverse over its natural domain $(-1, \infty)$. The inverse function is $f^{-1}(x) = \dfrac{1}{e^x - 1} - 1$, $x > 0$.

5. $(i)\ \dfrac{1}{2\sqrt{x(1-x)}}$ $\quad(ii)\ \dfrac{1}{\sqrt{-2x-x^2}}$ $\quad(iii)\ \dfrac{2\sin^{-1} x}{\sqrt{1-x^2}}$

$(iv)\ \dfrac{2x}{\sqrt{1-x^4}}$ $\quad(v)\ \dfrac{1}{2(1+x)\sqrt{x}}$ $\quad(vi)\ \dfrac{1}{(1+(\ln x)^2)x}$

6. (*i*) $\tan^{-1} x + C$ (*ii*) $-\tan^{-1}(\cos x) + C$

(*iii*) $\sin^{-1} e^x + C$ (*iv*) $\dfrac{1}{3}\tan^{-1}\left(\dfrac{x-2}{3}\right) + C$

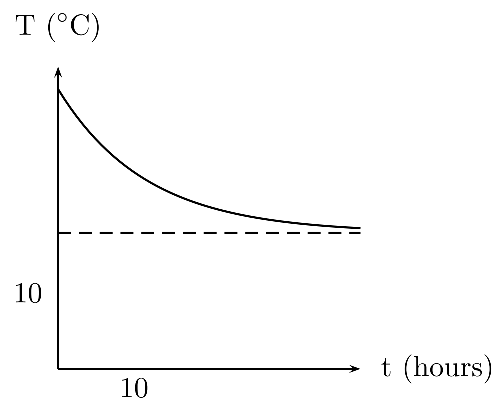
(*v*) $\sin^{-1}(x-1) + C$

2.9 Applications of calculus

3. (*i*) Let T be the temperature (°C) of the body at time t (hours), then

$$T(t) = 18 + 19e^{1/2\ln(16/19)t} \approx 18 + 19e^{-0.086t}.$$

(*ii*) The sketch of the function T(t)



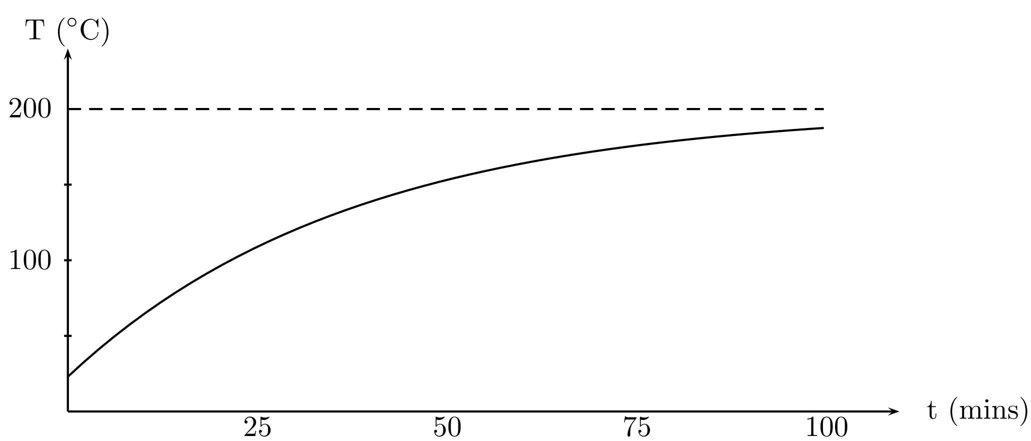
(*iii*) As time passes the temperature of the body decreases toward the room temperature, 18°C.

(*iv*) The murder occured 5.348 hours or approximately 5 hours 20 minutes before 4pm, and therefore at approximately 10:40am.

4. (*i*) Let T be the temperature (°C) of the potato at time t (minutes), then

$$T(t) = 200 - 177e^{1/30\ln(80/177)t} \approx 200 - 177e^{-0.0265t}.$$

(*ii*) The graph of the temperature of the potato of time



5. If we let T be the temperature of the metal rod in °C at time t minutes then we find the equation for the temperature over time is

$$T(t) = 30 + 1200e^{1/10 \ln(5/6)t} \approx 30 + 1200e^{-0.018t}.$$

Therefore, from this equation it is found that the metal rod will cool to 80°C approximately 174 minutes since the initial temperature measurement.

6. Firstly, we find that the equation for the temperature T (°C) over time t (minutes) is

$$T(t) = 20 - 16e^{1/25 \ln(10/16)t} \approx 20 - 16e^{-0.0188t}.$$

(i) After 50 minutes the temperature of the drink will be approximately 13.75°C.

(ii) The drink will reach 19.5°C after approximately 184 minutes.

8. Amplitude $= 2/3$; period $= 2\pi/3$; maximum speed $= 2$.

9. (i) Show that $\dfrac{d^2x}{dt^2} = -\dfrac{\pi^2}{4}x$.

(ii) Period $= 4$; amplitude $= 5$.

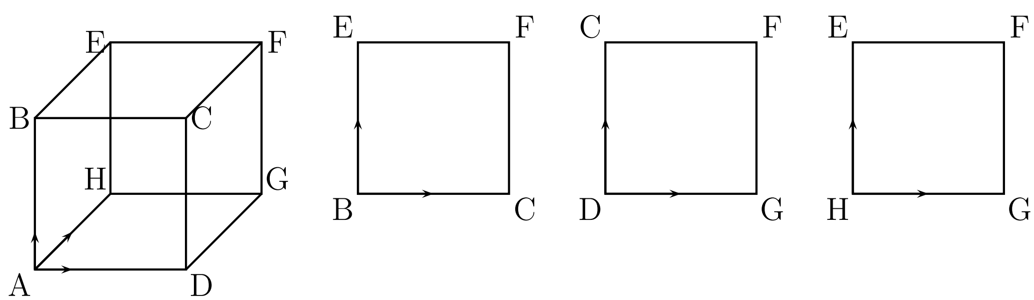
(iii) π^2

(iv) When $t = \dfrac{11}{3}$.

10. Period $= \pi$; amplitude $= 10$.

2.10 Counting and permutations

1. We want to start at A and end at F by going along 3 edges. From A there are 3 ways of going along the first edge, AB, AD, AH.



From each of B, D, H there are 2 ways of arriving at F (by going along 2 edges). By the product principle, there are $3 \times 2 = 6$ ways of going from A to F.

2. 256, 24

3. There are 10 ways to fill the first place, 9 the second, 8 the third and 7 the fourth. The answer is therefore $10 \times 9 \times 8 \times 7 = 5040$.

4. 12

5. There are 6 choices for the top and 7 for the bottom, so the number of different outfits is $6 \times 7 = 42$.

6. $\dfrac{7!}{4!}, \dfrac{11!}{7!}, \dfrac{3!}{6!}, \dfrac{13!}{10!3!}, \dfrac{n!}{3!(n-3)!}$

7. $\dfrac{r!}{(r-1)!} = \dfrac{r(r-1)!}{(r-1)!} = r,$

$\dfrac{r! + (r+1)!}{(r-1)!} = \dfrac{r! + (r+1)r!}{(r-1)!} = \dfrac{r!(r+2)}{(r-1)!} = r(r+2),$

$r! - (r-1)! = r(r-1)! - (r-1)! = (r-1)(r-1)!$

8. $2 \times 9!$

9. ${}^7P_5 = \dfrac{7!}{2!} = 7 \times 6 \times 5 \times 4 \times 3 = 2520$, ${}^8P_3 = \dfrac{8!}{5!} = 8 \times 7 \times 6 = 336$, ${}^{10}P_2 = \dfrac{10!}{8!} = 10 \times 9 = 90$, $\dfrac{{}^6P_4}{{}^3P_3} = \dfrac{6!}{2!} \times \dfrac{0!}{3!} = \dfrac{6 \times 5 \times 4 \times 3}{3 \times 2 \times 1} = 60$.

10. $(n+1)n(n-1)(n-2)$, $4n(n-1)(2n-1)$, $n(n-1)(n-2)\cdots(n-r+2)$, $n+1$

11. (i) ${}^{n-1}P_2 = \dfrac{(n-1)!}{(n-3)!} = (n-1)(n-2) = 56$. Thus $n-1 = 8$ and $n-2 = 7$, giving $n = 9$.

(ii) ${}^nP_4 = 90({}^{n-2}P_2)$ so $\dfrac{n!}{(n-4)!} = 90\dfrac{(n-2)!}{(n-4)!}$. Simplifying gives $n(n-1) = 90$ and so $n = 10$.

(iii) Since ${}^{n+1}P_5 = 72({}^{n-1}P_3)$, we have $\dfrac{(n+1)!}{(n-4)!} = 72\dfrac{(n-1)!}{(n-4)!}$.
Therefore $(n+1)n(n-1)! = 72(n-1)!$ and so $(n+1)n = 72$, or $n = 8$.

12. 6^4, 360

13. We first count the number of 4 digit numbers of the required type. The first digit must either be 5,8 or 9 so there are 3 choices for the first digit. The second digit can then be chosen in 5 ways, the third in 4 ways and the fourth in 3 ways. This gives $3 \times 5 \times 4 \times 3 = 180$ numbers. The 5 digit and 6 digit numbers are counted in a similar fashion. There are $5 \times 5 \times 4 \times 3 \times 2 = 600$ five digit numbers and $5 \times 5 \times 4 \times 3 \times 2 \times 1 = 600$ six digit numbers. The grand total is then $180 + 600 + 600 = 1380$. To count the 4 digit numbers which are multiples of 10, we note that the first digit can be chosen in 3 ways, the last in 1 way, the second in 4 ways and the third in 3 ways. This gives $3 \times 1 \times 4 \times 3 = 36$ such numbers. We count the 5 and 6 digit numbers in the same way. The grand total is then $36 + 5 \times 1 \times 4 \times 3 \times 2 + 5 \times 1 \times 4 \times 3 \times 2 \times 1 = 276$.

14. 600, 312, 240

15. A single ticket is between two nominated stations in a given order. There are ${}^{10}P_2 = \dfrac{10!}{8!} = 90$ such tickets.

16. $2 \times 5!$, $6 \times 4!$

17. The first digit must be 3 or 5, so there are 2 choices for the first digit. There are 5 choices for the second, 4 for the third and 3 for the fourth. The total number is therefore $2 \times 5 \times 4 \times 3 = 120$. If the number is even, then the fourth digit must be 2 or 8. There are 2 choices for the first, 2 for the fourth, 4 for second and 3 for third, giving $2 \times 2 \times 4 \times 3 = 48$ even numbers.

18. $5!$, $5 \times 4!$

19. There are $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 10^3 \times 26^3$ number plates.

2.11 Combinations and the binomial theorem

1. 15, 190, 3003

2. (i) Since $\dfrac{n!}{15!(n-15)!} = \dfrac{n!}{8!(n-8)!}$, we have $15!(n-15)! = 8!(n-8)!$, and so

$$15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 = (n-8)(n-9)(n-10)(n-11)(n-12)(n-13)(n-14)$$

giving $n-8 = 15$, and $n = 23$.

(ii) Since ${}^nC_3 = 8({}^nC_2)$, we have $\dfrac{n!}{3!(n-3)!} = \dfrac{8 \times n!}{2!(n-2)!}$ and so $3!(n-3)! = \dfrac{2!(n-2)!}{8}$. Thus $24 = n-2$ and so $n = 26$.

3. 28

4. There are ${}^{10}C_3 = \dfrac{10!}{3!7!} = \dfrac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$ ways of choosing the group of prefects. So after 120 days a particular group must serve again.

5. 126, 84, 56, 182

6. Once 3 men are chosen the remaining 7 automatically form a group of 7. So the number of ways is ${}^{10}C_3 = 120$.

7. 495, 7920

8. The number of subsets with 0 elements is 1 (the empty set). The number of subsets with 1 element is 4. The number of subsets with 2 elements is ${}^4C_2 = 6$ and the number of subsets with 3 elements is ${}^4C_3 = 4$. Therefore the total number of proper subsets is $1+4+6+4 = 15$.

9. $\{c, d, e\}$, $\{a, c\}$, 10, 10

10. ${}^{23}C_0 +^{23} C_1 +^{23} C_2 +^{23} C_3 = 1 + 23 + 23 \times 11 + 23 \times 11 \times 7 = 2048 = 2^{11}$.

11. (*i*) $1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$

(*ii*) $16 + 32x + 24x^2 + 8x^3 + x^4$

(*iii*) $27 - 54x + 36x^2 - 8x^3$

(*iv*) $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

(*v*) $x^5+5x^3+10x+10x^{-1}+5x^{-3}+x^{-5}$

(*vi*) $x^5-5x^3+10x-10x^{-1}+5x^{-3}-x^{-5}$

(*vii*) $x^6+9x^4y+27x^2y^2+27y^3$

(*viii*) $16+32x^3+24x^6+8x^9+x^{12}$

12. (*i*) The general term is ${}^7C_r(2x)^r=\dfrac{7!}{r!(7-r)!}2^rx^r$. The coefficient of x^4 is therefore $\dfrac{7!}{4!3!}2^4=35\times 2^4$.

(*ii*) The general term is ${}^5C_r(3x)^{5-r}(-2x^3)^r={}^5C_r3^{5-r}(-2)^rx^{5+2r}$. The coefficient of x^{11} is therefore (with $r=3$), ${}^5C_33^2(-2)^3=-720$.

(*iii*) The general term is ${}^7C_r(5x)^{7-r}(-3)^r$. The coefficient of x^4 occurs when $r=3$ and is therefore ${}^7C_35^4(-3)^3=-7\times 3^3\times 5^5$.

(*iv*) -7×2^6

(*v*) The general term is ${}^{10}C_r(x^2)^{10-r}(\dfrac{4}{x})^r={}^{10}C_r4^rx^{20-3r}$. The coefficient of x^{-1} occurs when $r=7$ and is therefore ${}^{10}C_74^7=3\times 4^8\times 10$

13. (*i*) 82 (*ii*) 416 (*iii*) 98

14. (*i*) The coefficient of x^6 in $(2+x)(1+x)^8$ is $2\times$ the coefficient of x^6 in $(1+x)^8$ plus the coefficient of x^5 in $(1+x)^8$. The required number is therefore $2({}^8C_6)+{}^8C_5=56+56=112$.

(*ii*) 196

(*iii*) The coefficient of x^8 in $(2-x^2)(3+x)^9$ is $2\times$ the coefficient of x^8 in $(3+x)^9$ minus the coefficient of x^6 in $(3+x)^9$. This number is $2\times{}^9C_8\times 3-{}^9C_6\times 3^3=-2214$

(*iv*) 140

15. $9x$

16. The coefficient of x^2 in $(2+x)(1+x)^n$ is the sum of the coefficient of x^2 in $2(1+x)^n$ and the coefficient of x in $(1+x)^n$. Therefore the required number is $2\binom{n}{2}+\binom{n}{1}=n^2$. The coefficient of x^3 in $(2+x)(1+x)^n$ is the sum of the coefficient of x^3 in $2(1+x)^n$ and the coefficient of x^2 in $(1+x)^n$. Therefore the required number is $2\binom{n}{3}+\binom{n}{2}=\dfrac{1}{6}n(n-1)(2n-1)$.

17. $2\binom{8}{r}-\binom{8}{r-1}$, $r=6$

18. $(1+kx)^n=1+{}^nC_1(kx)^1+{}^nC_2(kx)^2+\ldots\ldots=1+nkx+\dfrac{n(n-1)}{2}k^2x^2+\ldots\ldots$. So $nk=-12$ and $\dfrac{n(n-1)}{2}k^2=60$. Dividing the second equation by the first gives $(n-1)k=-10$ and then dividing the first by this one gives $\dfrac{n}{n-1}=\dfrac{12}{10}=\dfrac{6}{5}$. Thus $n=6$ and $k=-2$.

19. 743.7

20. We have ${}^nC_r = 455$ and ${}^nC_{r+1} = 1365$. Thus $\dfrac{n!}{r!(n-r)!} = 455$ and $\dfrac{n!}{(r+1)!(n-r-1)!} = 1365$. Dividing the second equation by the first gives $\dfrac{n-r}{r+1} = 3$, which shows that $n = 4r + 3$. We can now try various values of r. When $r = 0, 1, 2, 3, 4\ldots$ we get $n = 3, 7, 11, 15, 19\ldots$. By referring to the Pascal Triangle on page 2, it is clear that $n \neq 3, n \neq 7$. If $n = 11$ and $r = 2$ then ${}^{11}C_2 = 55$ so $n \neq 11$. If $n = 15$ and $r = 3$, we see that ${}^{15}C_3 = 455$. Therefore these numbers 455 and 1365 occur on the 16th row of the Pascal triangle (counting the single 1 as the first row).

Appendix A

Some standard integrals

$$\begin{aligned}
\int x^n \, dx &= \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1) \\
\int \frac{1}{x} \, dx &= \ln|x| + C \\
\int e^x \, dx &= e^x + C \\
\int \sin x \, dx &= -\cos x + C \\
\int \cos x \, dx &= \sin x + C \\
\int \sec^2 x \, dx &= \tan x + C \\
\int \frac{dx}{\sqrt{a^2 - x^2}} &= \sin^{-1} \frac{x}{a} + C \\
\int \frac{dx}{a^2 + x^2} &= \frac{1}{a} \tan^{-1} \frac{x}{a} + C
\end{aligned}$$

Appendix B

The Greek alphabet

α alpha
β beta
γ gamma (Upper case Γ)
δ delta (Upper case Δ)
ϵ epsilon
ζ zeta
η eta
θ theta (Upper case Θ)
ι iota
κ kappa
λ lambda (Upper case Λ)
μ mu
ν nu
ξ xi (Upper case Ξ)
o omicron
π pi (Upper case Π)
ρ rho
σ sigma (Upper case Σ)
τ tau
υ upsilon (Upper case Υ)
ϕ phi (Upper Case Φ)
χ chi
ψ psi (Upper case Ψ)
ω omega (Upper case Ω)