The inverse of a $n \times n$ matrix

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The $n \times n$ case

In the previous module we defined an inverse matrix and saw how to find the inverse of a 2×2 matrix, if it existed.

We will now find the inverse of a $n \times n$ matrix (if it exists), using Gaussian elimination.

We will illustrate this by finding the inverse of a 3×3 matrix.

First of all, we need to define what it means to say a matrix is in **reduced** row echelon form.

A matrix in reduced row echelon form is a row reduced matrix which has been simplified further by using the leading ones to eliminate the non-zero entries above them as well as below them.

Reduced row echelon form

A matrix is in reduced row echelon form if

1. the first nonzero entries of rows are equal to 1
2. the first nonzero entries of consecutive rows appear to the right
3. rows of zeros appear at the bottom
4. entries above and below leading entries are zero.

Here are some examples of matrices in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Using Gaussian elimination to find the inverse

Consider the matrix $B = \begin{bmatrix} 3 & 4 & -1 \\ 1 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix}$.

To find B^{-1}, if it exists, we augment B with the 3×3 identity matrix:

$$\left[\begin{array}{rrr|rrr} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \quad \text{ie} \quad [\,B\,|\,I\,].$$

The strategy is to use Gaussian elimination to reduce $[\,B\,|\,I\,]$ to reduced row echelon form. If B reduces to I, then $[\,B\,|\,I\,]$ reduces to $[\,I\,|\,B^{-1}\,]$.

B^{-1} appears on the right!

Reducing the matrix

Our first step is to get a 1 in the top left of the matrix by using an elementary row operation:

$$R_1 \longleftrightarrow R_2 \left[\begin{array}{rrr|rrr} \color{red}{1} & -1 & 1 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

Next we use the leading 1 (in red), to eliminate the nonzero entries below it:

$$R_2 - 3R_1 \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 7 & -4 & 1 & -3 & 0 \\ -1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 + R_1 \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 7 & -4 & 1 & -3 & 0 \\ 0 & 1 & 4 & 0 & 1 & 1 \end{array}\right].$$

Reducing the matrix

We now move down and across the matrix to get a leading 1 in the $(2, 2)$ position (in red):

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 7 & -4 & 1 & -3 & 0 \\ 0 & 1 & 4 & 0 & 1 & 1 \end{array}\right].$$

We can do this by $R_3 \longleftrightarrow R_2$ $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 & 1 \\ 0 & 7 & -4 & 1 & -3 & 0 \end{array}\right].$

Next, we use this leading 1 (in red) to eliminate all the nonzero entries above and below it:

$$\begin{array}{r} R_1 + R_2 \\ \\ R_3 - 7R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 0 & 2 & 1 \\ 0 & 1 & 4 & 0 & 1 & 1 \\ 0 & 0 & -32 & 1 & -10 & -7 \end{array}\right].$$

Reducing the matrix

Finally, we move down and across to the $(3, 3)$ position (in red):

$$\left[\begin{array}{ccr|crr} 1 & 0 & 5 & 0 & 2 & 1 \\ 0 & 1 & 4 & 0 & 1 & 1 \\ 0 & 0 & -32 & 1 & -10 & -7 \end{array}\right].$$

We make this entry into a leading 1 (in red) and use it to eliminate the entries above it:

$$-\tfrac{1}{32}R_3 \left[\begin{array}{ccc|rrr} 1 & 0 & 5 & 0 & 2 & 1 \\ 0 & 1 & 4 & 0 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{32} & \frac{10}{32} & \frac{7}{32} \end{array}\right]$$

$$\begin{array}{l} R_1 - 5R_3 \\ R_2 - 4R_3 \\ \end{array} \left[\begin{array}{ccc|rrr} 1 & 0 & 0 & \frac{5}{32} & \frac{14}{32} & -\frac{3}{32} \\ 0 & 1 & 0 & \frac{4}{32} & -\frac{8}{32} & \frac{4}{32} \\ 0 & 0 & 1 & -\frac{1}{32} & \frac{10}{32} & \frac{7}{32} \end{array}\right].$$

We have found B^{-1}

As $[\,B\,|\,I\,]$ reduced to $[\,I\,|\,B^{-1}\,]$

$$B^{-1} = \begin{bmatrix} \frac{5}{32} & \frac{14}{32} & \frac{-3}{32} \\ \frac{4}{32} & -\frac{8}{32} & \frac{4}{32} \\ -\frac{1}{32} & \frac{10}{32} & \frac{7}{32} \end{bmatrix} = \frac{1}{32}\begin{bmatrix} 5 & 14 & -3 \\ 4 & -8 & 4 \\ -1 & 10 & 7 \end{bmatrix}.$$

We can check that this matrix **is** B^{-1} by verifying that $BB^{-1} = B^{-1}B = I$.

$$\begin{aligned} B^{-1}B &= \frac{1}{32}\begin{bmatrix} 5 & 14 & -3 \\ 4 & -8 & 4 \\ -1 & 10 & 7 \end{bmatrix}\begin{bmatrix} 3 & 4 & -1 \\ 1 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix} \\ &= \frac{1}{32}\begin{bmatrix} 32 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Verify $BB^{-1} = I$ for yourself.