Further Probability Theory

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Contents

1 Introduction

This workbook introduces the formal study of probability as well as some techniques for solving problems.

Before starting, you will need to be familiar with all the concepts discussed in the two Mathematics Learning Centre workbooks "Introduction to Probability Theory" and "Counting Techniques". Try the problems at the end of thoses workbooks to ensure you have understood them.

1.1 How to use this book

You will not gain much by just reading this book. Have a pen and paper ready and try to work through the examples before reading their solutions. Do *all* the exercises. Solutions to the exercises are at the back of the book. It is important that you try hard to complete the exercises on your own, rather than refer to the solutions as soon as you are stuck.

1.2 Objectives

By the time you have worked through this workbook you should

- understand the basis of the study of probability,
- be able to represent diagrammatically experiments with two or more stages,
- be able to compute probabilities associated with tree diagrams,
- understand the difference between sampling with replacement and without replacement and be able to solve simple problems on these.

Assumed knowledge

1. Familiarity with the following: set theory, complementary events, incompatible events, conditional probability, independence, permutations and combinations.

2. The notation for the above concepts.

The above are covered in "Introduction to Probability Theory" and "Counting Techniques".

2 Review

1. When there are a finite number of equally likely outcomes of an experiment, the probability of an event A is

$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of possible outcomes}}.$$

For example, if a fair die is tossed, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

If A is the event 'an even number is obtained', then $A = \{2, 4, 6\}$ and $P(A) = \frac{3}{6} = \frac{1}{2}$.

2. The probability of an event happening lies between zero and one. The probability of the sample space is one while the probability of an impossible event is zero.

For example, the probability of obtaining 1, 2, 3, 4, 5 or 6 on a fair die is $P(S) = \frac{6}{6} = 1$.

The probability of obtaining -2 on this die is 0.

3. If two events A and B are mutually exclusive, that is, they cannot both occur at the same time, then the probability of A or B is $P(A \cup B) = P(A) + P(B)$.

We can think (loosely) of the probability of an event A as being the limiting value of the relative frequency of the event.

For example, if a fair coin is tossed, it is observed that as the experiment is carried out for a very large number of trials, the relative frequency of 'heads' is approximately $\frac{1}{2}$.

The 'definition' of probability given above, however, does not enable us to find the *exact* probability of an event. For example, if we toss a coin 100 times and it comes up heads 55 times, we cannot determine whether this is a biased coin with a probability of 0.55 of coming up heads, or whether it is a fair coin and the extra 5 heads were the result of chance. The same argument would apply no matter how often we tossed the coin.

For this reason, the 'relative frequency' definition of probability turns out to be unsatisfactory. We find it necessary to set up a mathematical model of probability by means of a set of axioms. This enables us to deal with random experiments where there are not necessarily a finite number of equally likely outcomes.

3 Axioms of probability

Let S be a sample space, E_i $(i = 1, 2 \cdots)$ be events of S, and P a function that associates with each event E, a real number $P(E)$. Then P is called the probability function and $P(E)$ is called the probability of the event E if the following axioms hold:

1. For every event E, $P(E) \geq 0$.

2. $P(S) = 1$.

3. If E_1, E_2, $\cdots$, are mutually exclusive events, then

$P(E_1 \cup E_2 \cup \cdots) = P(E_1) + P(E_2) + \cdots$.

In particular, if A and B are two mutually exclusive events (ie $A \cap B = \emptyset$) then

$P(A \cup B) = P(A) + P(B)$.

That is, if A and B are a pair of events which cannot occur together, the probability of A or B is the sum of their probabilities.

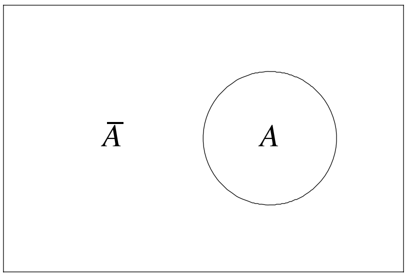
These axioms are really just a restatement, in formal mathematical language, of the properties listed in paragraphs 2 and 3 in the Review section.

It turns out we can deduce all the other properties of probability we need from just these three axioms. The examples below illustrate how this can be done.

Example 1

Show that if $\overline{A}$ is the complement of A in S, ie A and $\overline{A}$ are mutually exclusive and make up the whole sample space, then $P(\overline{A}) = 1 - P(A)$.

Solution



$A \cup \overline{A} = S$ so $P(A \cup \overline{A}) = 1$ by axiom 2.

That is, $P(A) + P(\overline{A}) = 1$ by axiom 3, since $A \cap \overline{A} = \emptyset$.

So, $P(\overline{A}) = 1 - P(A)$.

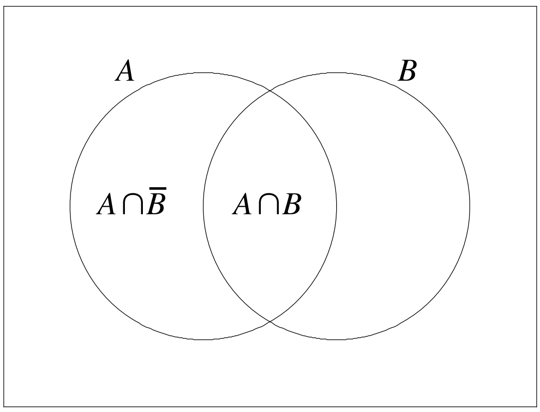
That is, the probability of the complement of an event is found by subtracting the probability of the event from one.

Example 2

Show that if A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Solution



We must first express $A \cup B$ as a union of *disjoint* sets, so that we can use axiom 3.

By inspection of the diagram,

$A \cup B = (A \cap \overline{B}) \cup B$ and these sets are disjoint.

Therefore, $P(A \cup B) = P(A \cap \overline{B}) + P(B)$. (1)

Now $A = (A \cap B) \cup (A \cap \overline{B})$ and these also are disjoint.

So, $P(A) = P(A \cap B) + P(A \cap \overline{B})$.

So, $P(A \cap \overline{B}) = P(A) - P(A \cap B)$.

Substituting this in (1), we get

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

We have shown that for any events A and B, the probability of at least one of them occurring equals the sum of their probabilities minus the probability that both A and B occur.

Note The results obtained in examples 1 and 2 are very useful. If you are not familiar with them already, you should make an effort to learn them.

3.1 Exercises 1–5

Try the following problems before looking at the worked solutions. You may find it helpful to refer to the previous two workbooks mentioned in the introduction.

1. Two cards are drawn at random from a pack of 52 cards. Find the probability that:

a. one is a club and one is a spade,

b. both are hearts.

2. A coin is weighted so that heads (H) is twice as likely to appear as tails (T). Find $P(H)$ and $P(T)$.

3. In a manufacturing process, the probability that a component is defective is $\frac{1}{8}$. If 40 components are selected at random, find the probability that:

a. all the components are good,

b. exactly one is defective,

c. at least two are defective.

4. In a certain college 60% of the students pass mathematics, 70% of them pass chemistry and 50% of them pass both mathematics and chemistry. A student is selected at random.

a. If she passed mathematics, what is the probability that she passed chemistry?

Hint: The conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

b. If she passed chemistry, what is the probability that she passed maths?

c. What is the probability that she passed maths or chemistry?

d. Are the events M 'the student passes maths' and C 'the student passes chemistry' statistically independent?

Hint: A and B independent means that $P(A|B) = P(A)$.

Therefore, $P(A \cap B) = P(A)P(B)$.

5. A couple has 3 children. Let A be the event 'they have at least one child of each sex' and B the event 'they have at most one boy'. Show that A and B are statistically independent.

4 Tree diagrams

Many problems in probability which are 'multiple stage' experiments may be done by using tree diagrams. We shall discuss a few of these.

4.1 Sampling without replacement

Example 3

A box contains 4 red marbles and 3 white ones. A marble is drawn at random and not replaced. A second marble is then drawn from the box. What is the probability that

a. two white marbles are drawn,

b. a red marble is drawn first and then a white one,

c. a red marble and a white one are drawn (in any order).

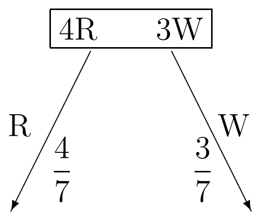
Solution

Let R be the event 'a red marble is drawn'.

Let W be the event 'a white marble is drawn'.

On the first trial we may draw a red marble with probability $\frac{4}{7}$ or a white marble with probability $\frac{3}{7}$.

This may be represented diagrammatically as follows:



Now if a red is drawn on the first trial there are 3 red and 3 white marbles left.

On the second trial a red marble or a white marble may be drawn.

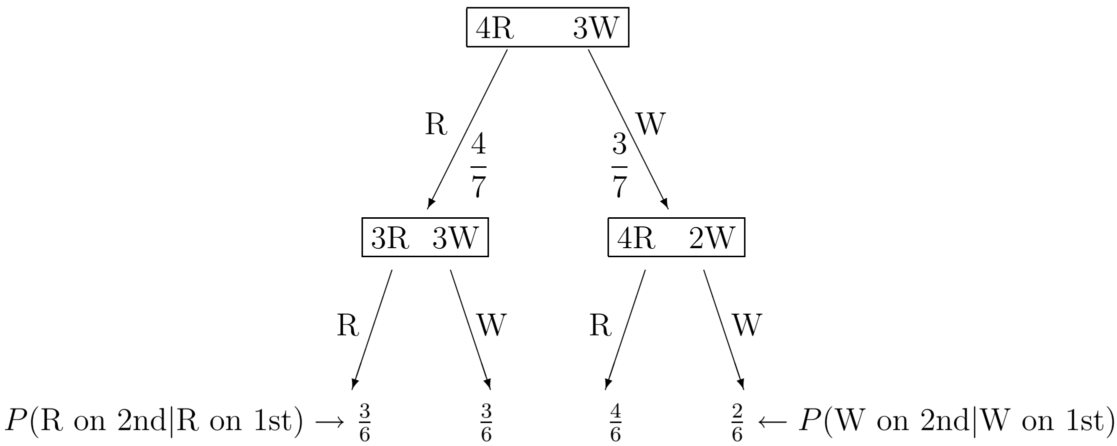
Hence the probability of a red on the second trial *given* that a red was drawn on the first trial is $\frac{3}{6}$ and similiarly the probability of white on the second trial given red on the first is $\frac{3}{6}$.

If a white marble is drawn first time, then there are 4 red marbles and 2 white marbles left.

Hence $P(\text{white on second trial}|\text{white on first trial}) = \frac{2}{6}$.

Hence $P(\text{red on second trial}|\text{white on first trial}) = \frac{4}{6}$.

We can represent these on the tree diagram as shown.



Notes

1. The four probabilities shown in the 2nd stage of the tree are conditional probabilities.

2. Since $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, for $P(B) \neq 0$, we have $P(A \cap B) = P(B)P(A|B)$.

We use this to find the probabilities of these four events:

'red on first draw and red on second draw',

'red on first draw and white on second',

'white on first draw and red on second', and

'white on first draw and white on second'.

To find the probabilities of each of these events we simply multiply together the probabilities for the two corresponding branches.

a. The probability of drawing two white marbles is

$$P(\text{white on 1st and white on 2nd}) = P(\text{white on 1st}) \times P(\text{white on 2nd}|\text{white on 1st}).$$

Thus $P(\text{two white marble drawn}) = \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$.

b.

$$
\begin{aligned}
P(\text{a red and } \textbf{then} \text{ a white}) &= P(RW) \\
&= P(R\text{on 1st}) \times P(W\text{on 2nd}|R \text{ on 1st}) \\
&= \frac{4}{7} \times \frac{3}{6} \\
&= \frac{2}{7}.
\end{aligned}
$$

c.

$$
\begin{aligned}
P(\text{a red and a white}) &= P(RW\text{or } WR) \\
&= P(RW) + P(WR) \quad \text{since these are mutually exclusive} \\
&= \frac{4}{7} \times \frac{3}{6} + \frac{3}{7} \times \frac{4}{6} \\
&= \frac{4}{7}.
\end{aligned}
$$

Exercise 6

In Example 3

a. Find the probability that two reds are drawn.

b. Are the events ‘white on the second draw’ and ‘red on the first draw’ independent?

4.2 Sampling with replacement

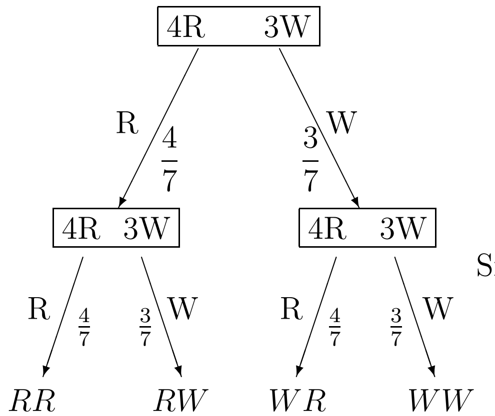
Example 4

Suppose in Example 3 a marble is drawn at random and then replaced in the box. A second marble is drawn at random.

a. Find the probability that two white marbles are drawn.

b. Find the probability that at least one white marble is drawn.

Solution



Since marble is replaced

a. P(white on 1st and white on 2nd) $= \frac{3}{7} \times \frac{3}{7} = \frac{9}{49}$.

b.

$$\begin{aligned}
\text{P(at least one white)} &= \text{P(WR or RW or WW)} \\
&= \text{P(WR)} + \text{P(RW)} + \text{P(WW)} \\
&= \frac{3}{7} \times \frac{4}{7} + \frac{4}{7} \times \frac{3}{7} + \frac{3}{7} \times \frac{3}{7} \\
&= \frac{33}{49}.
\end{aligned}$$

Exercise 7

Are all events ‘white on the second draw’ and ‘red on the first draw’ statistically independent?

Example 5

A set of 10 equally matched toy racing cars are raced twice. If 5 of the cars are BMWs, 4 are Lamborghinis and 1 is a Renault model, what is the probability that:

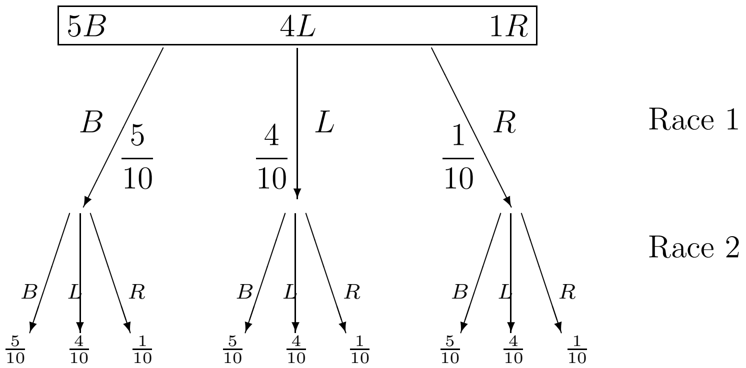
a. both races are won by the BMWs,

b. both races will be won by a car of the same make.

Solution

This problem is equivalent to a sampling with replacement problem since all the cars race each time and the same car can win twice.

The tree diagram is as follows:



In the above diagram, B is the event ‘a BMW wins the race’, L is the event ‘a Lamborghini wins’ and R the event ‘the Renault wins’.

a. $P(B \cap B) = \frac{5}{10} \times \frac{5}{10} = \frac{1}{4}$.

b.

$$\begin{aligned}
P(B \cap B) + P(R \cap R) + P(L \cap L) &= (\frac{5}{10})^2 + (\frac{4}{10})^2 + (\frac{1}{10})^2 \\
&= \frac{42}{100} \\
&= \frac{21}{50}.
\end{aligned}$$

Exercise 8

In the Example 5:

a. Find the probability that a BMW wins the first race and a Lamborghini the second.

b. Find the probability that the Renault wins one race.

5 Binomial probability

Consider the problem:

A fair die is rolled 5 times. What is the probability of getting exactly three 'ones'?

Let us first look at one roll of the die. The probability of successfully getting a 'one' is $\frac{1}{6}$, while the probability of failing to get a 'one' is $\frac{5}{6}$.

Now the outcome of any one roll of the die will not affect the outcome of any of the other four rolls – the trials are independent.

Thus the probability of getting this sequence of outcomes: 1, 1, 1, not a one, not a one is $(\frac{1}{6})^3(\frac{5}{6})^2$. We are multiplying together the probabilities for each outcome.

As we are not interested in what numbers come up, other than 'one' or 'not one', let us denote by S (for success) the event that a 'one' comes up and F (failure) the event that any other number comes up.

The possible ways of obtaining exactly 3 successes in 5 trials are:

$S \quad S \quad S \quad F \quad F$

$S \quad S \quad F \quad S \quad F$

$S \quad S \quad F \quad F \quad S$

$S \quad F \quad S \quad S \quad F$

$S \quad F \quad S \quad F \quad S$

$S \quad F \quad F \quad S \quad S$

$F \quad S \quad S \quad S \quad F$

$F \quad S \quad S \quad F \quad S$

$F \quad S \quad F \quad S \quad S$

$F \quad F \quad S \quad S \quad S$

ten ways in all.

Of course these ten sequences are simply the number of ways of choosing 3 positions for S out of a possible 5, ie $^5C_3$. Each sequence has the same probability of occuring, namely $(\frac{1}{6})^3(\frac{5}{6})^2$.

So the answer to the question is $^5C_3(\frac{1}{6})^3(\frac{5}{6})^2 = 0.032$.

The coefficient $^5C_3$ is called the binomial coefficient.

To understand why we call it a binomial coefficient, refer to the Mathematics Learning Centre workbook "Counting Techniques", and for more on Binomial probability to the workbook "Binomial Distribution".

Questions on binomial probability have the following characteristics:

a. An experiment is repeated a finite number, n, times (called n trials);

b. Each trial results in one of two possibilities, which we may denote 'success' and 'failure';

c. The trials are independent; and

d. The probability of 'success', p, is the same for each trial.

If these conditions are fulfilled, then the chance that an event will occur exactly k times out of n is given by the formula:

$$^nC_k p^k (1-p)^{n-k}.$$

Exercise 9

Show that this formula is true by generalising the reasoning for the die example given above.

Exercise 10

Determine whether each of the following problems can be done using binomial probability and if so solve the problem using this method.

a. A fair coin is tossed 8 times. What is the chance of getting exactly 3 tails?

b. A card is drawn at random and replaced three times from a pack of cards. Find the probability that exactly two hearts are drawn.

c. A box contains 4 red marbles and 3 white ones. A marble is drawn at random and not replaced. A second marble is drawn. Find the probability that two reds are drawn. (This is exercise 6(a).)

d. Exercise 3.

6 Worksheet

Try these problems to see if you have understood this workbook. The solutions are given in the next section.

1. For the simple toss of a die, let A be the event 'the number 4 appears', B the event 'an odd number appears', and C the event 'a prime number appears'.

a. The die is tossed 40 times. The following table shows the frequency with which each number appeared.

Number	1	2	3	4	5	6
Frequency	8	6	9	7	9	1

Find the relative frequency of the events A, B and C.

b. Assuming the die is fair, find $P(A)$, $P(B)$ and $P(C)$.

2. Three horses, Flash, Speed and Zoom are in a race. Flash is three times as likely to win as Speed and Speed is twice as likely to win as Zoom. Assuming a draw is impossible, find their respective probabilities of winning.

3. A box contains twelve items, of which four are defective. Two items are chosen at randon from the box. Find the probability that one item is defective and one is good, and the probability that both are defective.

4. In a certain class, 40% of the students play tennis, 25% of them play squash and 15% of them play both tennis and squash. A person is selected at random from the class.

a. If she plays tennis, what is the probability that she also plays squash?

b. If she plays squash, what is the probability that she does not play tennis?

5. Let A and B be events with $P(A) = \frac{1}{5}$, $P(A \cup B) = \frac{1}{3}$ and $P(B) = x$.

a. Find x if A and B are mutually exclusive.

b. Find x if A and B are independent.

6. Box A contains 7 cards numbered 1 through 7 and box B contains 5 cards numbered 1 through 5. A box is chosen at random and a card is drawn. If the number on the card is even, find the probability that the card came from box B. (Draw a tree diagram.)

7. An urn contains 3 red marbles and 6 blue ones. A marble is drawn from the urn and a marble of the other colour is put into the urn. A second marble is drawn from the urn.

a. Draw a tree diagram for the above.

b. Find the probability that the second marble drawn is blue.

8. A box contains one red marble and four green ones. Five draws are made at random with replacement.

a. What is the chance that exactly two of the marbles drawn are red?

b. What is the probability that at least one is green?

7 Solutions

7.1 Solutions to exercises

Remember, there may be many different ways to do a problem. The methods given here are just suggestions.

1. The number of ways of choosing 2 cards from 52 is ${}^{52}C_2 = \dfrac{52.51}{2} = 26 \times 51$.

a. There are 13 clubs and 13 spades in the pack, thus there are 13×13 ways of choosing a spade and a club.
Therefore,

$$P(\text{one club and one spade}) = \frac{13 \times 13}{26 \times 51} = \frac{13}{102}.$$

b. There are ${}^{13}C_2$ ways of choosing two hearts out of 13 in the pack.
Therefore,

$$P(\text{both cards are hearts}) = \frac{{}^{13}C_2}{{}^{52}C_2} = \frac{\frac{13.12}{2}}{\frac{52.51}{2}} = \frac{3}{51} = \frac{1}{17}.$$

2. By axiom (ii), $P(H) + P(T) = 1$.

Now, $P(H) = 2P(T)$, and therefore $2P(T) + P(T) = 3P(T) = 1$.

That is, $P(T) = \frac{1}{3}$ and $P(H) = \frac{2}{3}$.

3. Suppose that one component is selected at random.

Let D be the event 'the component selected is defective', and G the event 'the component selected is good'.

Then $P(D) = \frac{1}{8}$ so $P(G) = 1 - \frac{1}{8} = \frac{7}{8}$.

a. P(all 40 components are good)
$= P$(1st component is good **and** 2nd is good and $\cdots$ and 40th is good).

Since the probability of selecting a defective component is always $\frac{1}{8}$, the probability of getting a good one is always $\frac{7}{8}$.

This means that each selection is **independent** of all the others. Hence,

$$P(\text{all 40 are good}) = (\frac{7}{8})^{40} = 0.0048.$$

[Recall that, when events A and B are independent, $P(A \cap B) = P(A)P(B)$.]

b. Let us first find the probability of the 40 randomly selected components being chosen in the following order:
$D \quad G \quad G \quad G \cdots G$. That is, there is one defective compoment (the first one) and 39 good ones.
The probability of this is $(\frac{1}{8})(\frac{7}{8})^{39}$.
But the event 'exactly one defective component out of 40' can occur in
${}^{40}C_1 = 40$ ways.

$D \quad G \quad G \quad G \cdots G$ or
$G \quad D \quad G \quad G \cdots G$ or
$G \quad G \quad D \quad G \cdots G$ etc.

All of these have probability $(\frac{1}{8})(\frac{7}{8})^{39}$.
Hence,

$$P(\text{1 defective component out of 40}) = 40(\frac{1}{8})(\frac{7}{8})^{39} = 0.027.$$

c. P(at least two defective components)
$= 1 - P$(no defective or 1 defective) using complementary events
$= 1 - (0.0048 + 0.027) = 0.968$.

4. Let M be the event 'the student passes maths' and C be the event 'the student passes chemistry'.

Then $P(M) = 0.6$, $P(C) = 0.7$ and $P(M \cap C) = 0.5$.

a. Find $P(C|M)$.

$$\begin{aligned} P(C|M) &= \frac{P(C \cap M)}{P(M)} \\ &= \frac{0.5}{0.6} \\ &= \frac{5}{6}. \end{aligned}$$

b.

$$\begin{aligned} P(M|C) &= \frac{P(M \cap C)}{P(C)} \\ &= \frac{0.5}{0.7} \\ &= \frac{5}{7}. \end{aligned}$$

c. $P(M \cup C) = P(M) + P(C) - P(M \cap C) = 0.6 + 0.7 - 0.5 = 0.8$

d. M and C are statistically independent if and only if
$P(M \cap C) = P(M)P(C)$.
Now $P(M \cap C) = 0.5$ and $P(M)P(C) = 0.6 \times 0.7 = 0.42$, therefore they are not independent.

5. For 3 children, the sample space S is

$S = \{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG\}$,

$A = \{BBG, BGB, GBB, BGG, GBG, GGB\}$,

$B = \{BGG, GBG, GGB, GGG\}$ and

$A \cap B = \{BGG, GBG, GGB\}$.

Hence $P(A) = \frac{6}{8}$, $P(B) = \frac{4}{8}$ and $P(A \cap B) = \frac{3}{8}$.

Therefore, $P(A)P(B) = \frac{6}{8} \times \frac{4}{8} = \frac{3}{8} = P(A \cap B)$,

So A and B are statistically independent.

Notice that A and B satisfy the rule for statistical independence but in our everyday understanding of the word 'independent', A and B do not appear to be independent events. Therefore, always check the rule for statistical independence – it does not always agree with our intuitive understanding, as this example shows.

6. **a.** $P(RR) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$.

b.

$$\begin{aligned} P(W \text{on 2nd draw}) &= P(RW \text{or } WW) \\ &= P(RW) + P(WW) \\ &= \frac{4}{7} \times \frac{3}{6} + \frac{3}{7} \times \frac{2}{6} \\ &= \frac{3}{7}. \end{aligned}$$

$P(R \text{ on 1st draw}) = \frac{4}{7}$.
Now $P(R \text{ on 1st draw and } W \text{on 2nd}) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$, and
$P(R \text{ on 1st draw}) \times P(W \text{on 2nd}) = \frac{4}{7} \times \frac{3}{7} = \frac{12}{49}$.
Since these expressions are not equal, we conclude that the events are not independent.

Alternatively, we can show that $P(W \text{on 2nd}|R \text{ on 1st}) \neq P(W \text{on 2nd})$. Hence using the definition of independence we see that these two events are not independent.

7.

$$\begin{aligned} P(W \text{on 2nd}) &= P(RW) + P(WW) \\ &= \frac{4}{7} \times \frac{3}{7} + \frac{3}{7} \times \frac{3}{7} \\ &= \frac{21}{49} \\ &= \frac{3}{7}. \end{aligned}$$

$P(W \text{on 2nd}|R \text{ on 1st}) = \frac{3}{7}$ from the tree diagram.

Thus the events are statistically independent – which is to be expected.

8. a. $P(\text{BMW wins first race and Lamborghini wins 2nd race}) = \frac{5}{10} \times \frac{4}{10} = \frac{1}{5}$.

b.

$$\begin{aligned} P(\text{Renault wins exactly one race}) &= P(RB) + P(RL) + P(BR) + P(LR) \\ &= \frac{1}{10} \times \frac{5}{10} + \frac{1}{10} \times \frac{4}{10} + \frac{5}{10} \times \frac{1}{10} + \frac{4}{10} \times \frac{1}{10} \\ &= \frac{18}{100} \\ &= \frac{9}{50}. \end{aligned}$$

9. There are n trials and the probability of success on any one trial is p. The probability of failure is $(1-p)$.

Consider any one sequence of trials having k successes, say the first k trials have outcome S and the remaining $(n-k)$ trials have outcome F:

$S\ S\ S\ \cdots S\ F\ F\ F\ \cdots F.$

The probability of this sequence occurring is $p^k(1-p)^{n-k}$ as the trials are independent and so we multiply the probabilities for each outcome.

Now the number of different sequences having k places out of a possible n is ${}^nC_k$. Each of these sequences has the same probability of occurring.

Hence, $P(k \text{ successes in } n \text{ trials}) = {}^nC_k p^k(1-p)^{n-k}$.

10. a. This satisfies the required conditions. We have $n = 8$, the trials are independent and each trial has two possible outcomes. If we let S be the event 'a tail appears', then $p = P(S) = \frac{1}{2}$.
$P(3 \text{ tails}) = {}^8C_3(\frac{1}{2})^3(\frac{1}{2})^5 = 0.219$.

b. Again, the required conditions are satisfied.
Let S be the event 'a heart is drawn'.
Then $n = 3$, $P(S) = p = \frac{13}{52} = \frac{1}{4}$ and
$P(2 \text{ hearts drawn}) = {}^3C_2(\frac{1}{4})^2(\frac{3}{4}) = 0.141$.

c. The trials in this experiment are not independent. The probability that a red marble is drawn first time is $\frac{4}{7}$ but the probability that a red is drawn the second time depends on the outcome of the first trial.

d. This problem is, of course, a binomial probability problem. There are 40 trials, and they are independent. If we (arbitrarily) decide that 'success' is 'getting a defective component' then $P(S) = p = \frac{1}{8}$.

$$\begin{aligned}
P(\text{all the components are good}) &= P(\text{no defectives}) \\
&= {}^{40}C_0(\frac{1}{8})^0(\frac{7}{8})^{40} \\
&= (\frac{7}{8})^{40} \\
&= 0.0048.
\end{aligned}$$

$P(\text{exactly one component is defective}) = {}^{40}C_1(\frac{1}{8})(\frac{7}{8})^{39} = 0.027$ as before.

$P(\text{at least two are defective}) = 1 - (0.0048 + 0.027) = 0.968$ as before.

7.2 Solutions to worksheet

1. a.

$$\text{Relative frequency of } A = \frac{7}{40} = 0.175,$$

$$\begin{aligned}\text{Relative frequency of } B &= \frac{26}{40} = 0.65,\\ \text{Relative frequency of } C &= \frac{24}{40} = 0.6.\end{aligned}$$

b. $P(A) = \frac{1}{6} = 0.16,$
$P(B) = \frac{3}{6} = 0.5$ since there are 3 odd numbers,
$P(C) = \frac{3}{6} = 0.5$ since there are 3 primes: 2, 3 and 5.

2. Since we are assuming that a draw is impossible, the events 'Flash wins', 'Speed wins' and 'Zoom wins' are mutually exclusive.

Let $P(\text{Zoom wins}) = x$.

Then $P(\text{Speed wins}) = 2x$ and $P(\text{Flash wins}) = 6x$.

Then $x + 2x + 6x = 9x = 1$ using axioms (ii) and (iii).

Hence, $x = \frac{1}{9}$ and $P(\text{Zoom wins}) = \frac{1}{9}$, $P(\text{Speed wins}) = \frac{2}{9}$ and $P(\text{Flash wins}) = \frac{6}{9}$.

3. The total number of ways of choosing 2 items from 12 is ${}^{12}C_2$.

The number of ways of choosing one defective item from 4 defective ones is ${}^{4}C_1$, while the number of ways of choosing 1 good item from the 8 good ones is ${}^{8}C_1$. Hence,

$$P(\text{one defective and one good}) = \frac{{}^{4}C_1 \times {}^{8}C_1}{{}^{12}C_2} = \frac{16}{33}.$$

$$P(\text{both defective}) = \frac{{}^{4}C_2}{{}^{12}C_2} = \frac{1}{11}.$$

Note: This question can also be done using tree diagrams.

4. Let T be the event 'the person selected plays tennis' and S the event 'the person selected plays squash'.

Then $P(T) = \frac{40}{100}$, $P(S) = \frac{25}{100}$ and $P(T \cap S) = \frac{15}{100}$.

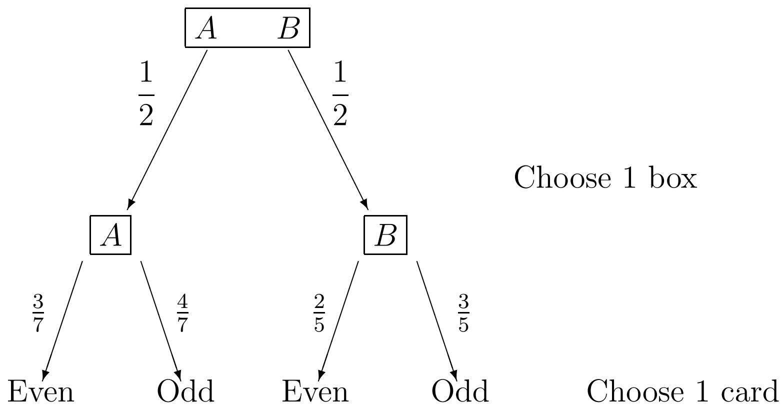
a. We want $P(S|T)$.
Now $P(S|T) = \dfrac{P(S \cap T)}{P(T)} = \dfrac{\frac{15}{100}}{\frac{40}{100}} = \dfrac{3}{8}.$

b. We want $P(\overline{T}|S) = \dfrac{P(\overline{T} \cap S)}{P(S)}.$
Now 10% of the students (25% − 15%) play squash but not tennis,
so, $P(\overline{T} \cap S) = \frac{10}{100}$.
Therefore, $P(\overline{T}|S) = \dfrac{P(\overline{T} \cap S)}{P(S)} = \dfrac{\frac{10}{100}}{\frac{25}{100}} = \dfrac{10}{25} = \dfrac{2}{5}.$

5. a. If A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$.
ie $\frac{1}{3} = \frac{1}{5} + x$.
Therefore $x = \frac{2}{15}$.

b. If A and B are independent, then $P(A \cap B) = P(A)P(B) = \frac{1}{5}x$.
Now by Example 2, we have $P(A) + P(B) - P(A \cap B) = P(A \cup B)$.
ie $\frac{1}{5} + x - \frac{1}{5}x = \frac{1}{3}$
ie $x = \frac{1}{6}$.

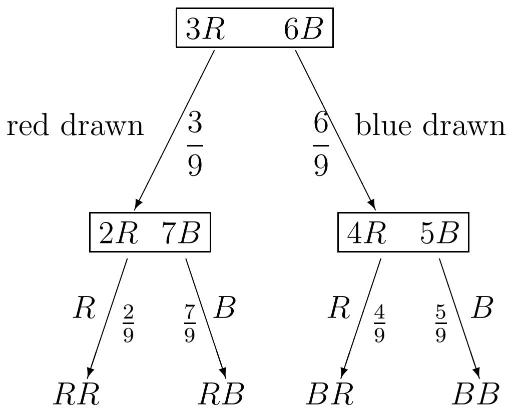
6.



Let A be the event 'box A is chosen', then $P(A) = \frac{1}{2}$.
Let B be the event 'box B is chosen', then $P(B) = \frac{1}{2}$.
Let E be the event 'an even card is chosen', then $P(E) = \frac{1}{2} \times \frac{3}{7} + \frac{1}{2} \times \frac{2}{5}$.
Let O be the event 'an odd card is chosen', then $P(O) = \frac{1}{2} \times \frac{4}{7} + \frac{1}{2} \times \frac{3}{5}$.
We want

$$P(B|E) = \frac{P(B \cap E)}{P(E)} = \frac{\frac{1}{2} \times \frac{2}{5}}{\frac{1}{2} \times \frac{3}{7} + \frac{1}{2} \times \frac{2}{5}} = \frac{14}{29}.$$

7.



$$\begin{aligned} P(\text{second marble is blue}) &= P(RB) + P(BB) \\ &= \frac{3}{9} \times \frac{7}{9} + \frac{6}{9} \times \frac{5}{9} \\ &= \frac{17}{27}. \end{aligned}$$

8. Let us denote by S the event that 'a red marble is drawn'. Then we can see that the conditions for binomial probability are satisfied, as the trials are independent with probability of success, p, being $\frac{1}{5}$ for each trial. (This would not be the case if the marbles drawn were not replaced.)

a.

$$^5C_2(\frac{1}{5})^2(\frac{4}{5})^3 = 0.2.$$

b. The probability that at least one marble is green is $1 -$ probability of no greens.

Now $P(\text{no greens}) = P(5 \text{ reds}) = (\frac{1}{5})^5 = 0.00032$.

Therefore, $P(\text{at least one green}) = 1 - 0.00032 = 0.9997$, almost a certainty.

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