Introduction to Exponents and Logarithms

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1 Exponents

1.1 Introduction

Whenever we use expressions like 7^3 or 2^5 we are using exponents.

The symbol 2^5 means $\underbrace{2 \times 2 \times 2 \times 2 \times 2}_{\text{5 factors}}$. This symbol is spoken as 'two raised to the power five', 'two to the power five' or simply 'two to the five'. The expression 2^5 is just a shorthand way of writing 'multiply 2 by itself 5 times'. The number 2 is called the *base*, and 5 the *exponent*.

Similarly, if b is any real number then b^3 stands for $b \times b \times b$. Here b is the base, and 3 the exponent.

If n is a whole number, b^n stands for $\underbrace{b \times b \times \cdots \times b}_{n \text{ factors}}$. We say that b^n is written in *exponential form*, and we call b the base and n the exponent, power or index.

Special names are used when the exponent is 2 or 3. The expression b^2 is usually spoken as 'b squared', and the expression b^3 as 'b cubed'. Thus 'two cubed' means $2^3 = 2 \times 2 \times 2 = 8$.

1.2 Exponents with the Same Base

We will begin with a very simple definition. If b is any real number and n is a positive integer then b^n means b multiplied by itself n times. The rules for the behaviour of exponents follow naturally from this definition.

First, let's try multiplying two numbers in exponential form. For example

$$
\begin{aligned}
2^3 \times 2^4 &= (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2) \\
&= \underbrace{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}_{\text{7 factors}} \\
&= 2^7 \\
&= 2^{3+4}.
\end{aligned}
$$

Examples like this suggest the following general rule.

Rule 1: $b^n \times b^m = b^{n+m}$.

That is, to *multiply* two numbers in exponential form (with the same base), we *add* their exponents.

Let's look at what happens when we divide two numbers in exponential form. For example,

$$
\begin{aligned}
\frac{3^6}{3^4} &= \frac{3 \times 3 \times 3 \times 3 \times 3 \times 3}{3 \times 3 \times 3 \times 3} \\
&= 3 \times 3 \times \frac{3 \times 3 \times 3 \times 3}{3 \times 3 \times 3 \times 3} \\
&= 3 \times 3 \\
&= 3^2 \\
&= 3^{6-4}.
\end{aligned}
$$

This leads us to another general rule.

Rule 2: $\frac{b^n}{b^m} = b^{n-m}$.

In words, to *divide* two numbers in exponential form (with the same base) , we *subtract* their exponents.

We have not yet given any meaning to negative exponents, so n must be greater than m for this rule to make sense. In a moment we will see what happens if n is not greater than m.

Now look at what happens when a number in exponential form is raised to some power. For example,

$$\begin{aligned}(2^2)^3 &= (2 \times 2) \times (2 \times 2) \times (2 \times 2) \\ &= 2^6 \\ &= 2^{2 \times 3}.\end{aligned}$$

This suggest another general rule.

Rule 3: $(b^m)^n = b^{mn}$

That is, to raise a number in exponential form to a power, we *multiply* the exponents.

Examples

$5^2 \times 5^4 = 5^{2+4} = 5^6 = 15625$

$\dfrac{2.5^4 \times 2.5^3}{2.5} = 2.5^{4+3-1} = 2.5^6 = 244.140625$

$(\dfrac{\pi^5}{\pi^3})^3 = (\pi^{5-3})^3 = (\pi^2)^3 = \pi^{2\times 3} = \pi^6 \approx 961.389$

$\dfrac{3^{n+4}}{3^{m+1}}$ 'simplifies' to 3^{n-m+3}.

$\dfrac{x^3y^4}{xy^2}$ simplifies to x^2y^2.

Exercises
Evaluate the following expressions using a calculator where necessary.

1. 3^4 **2.** $4^2 \times 3^3$ **3.** $\frac{1.5^7}{1.5^4}$ **4.** $(2.7^2)^3$ **5.** $(3 + 2.2^3)^4$

Simplify these, or at least change them around a bit.

6. $\frac{2^{2x+y}}{2^x}$ **7.** $\frac{x^7x^{9.5}}{x^{1.2}}$ **8.** 4^{x+3y}

9. $(3^x)^z$ **10.** $\frac{3^{x+2z}}{3^{\frac{z}{4}}}$

Until now we have only considered exponents which are positive integers, such as 7 or 189. Our intention is to extend this notation to cover exponents which are not necessarily positive integers, for example -5, or $\frac{113}{31}$, or numbers such as $\pi \approx 3.14159$. Just as we can make sense of expressions like 5^{189}, we want to be able to make sense of expressions such as $5^{\frac{113}{31}}$. But more than this, we want to make sense of these expressions in such a way that rules 1, 2 and 3 remain valid. It is not at all obvious how we should interpret an expression $5^{\frac{113}{31}}$. It does not really make sense to think of it as 5 multiplied by itself $\frac{113}{31}$ times.

Our plan is this: if we want rules 1, 2 and 3 to hold for general exponents then we will try defining expressions like $5^{\frac{113}{31}}$ to be whatever they must be in order that rules 1, 2 and 3 remain valid. In other words, we will insist that rules 1, 2 and 3 remain valid for these more general exponents, and hope that this requirement will tell us what the definitions of expressions like $5^{\frac{113}{31}}$ must be.

Let us begin by extending the notation to include an exponent equal to 0. We want to make sense of the expression b^0 in such a way that rules 1, 2 and 3 hold. What happens to rule 2 when $n = m$? Rule 2 gives

$$\begin{aligned} \frac{b^n}{b^n} &= b^{n-n} \quad \text{or} \\ 1 &= b^0. \end{aligned}$$

Until now we have not attached any meaning to the expression b^0. It doesn't make sense to talk about a number being multiplied by itself 0 times. However, if we want rule 2 to continue to be valid when $n = m$ then we must *define* the expression b^0 to mean the number 1.

If $b \neq 0$ then we define b^0 to be equal to 1. We do not attempt to give any meaning to the expression 0^0. It remains undefined.

Using this definition we can check that rules 1 and 3 also remain valid. For example, to check that rule 1 still holds, if n is a whole number and $m = 0$ then rule 1 gives

$$b^n \times b^0 = b^n$$

which is okay because $b^0 = 1$.

To be strictly correct we should also check that rule 1 remains valid in the case that $m = 0$ and $n = 0$. You should check that this is true and that rule 3 also remains valid under this definition of b^0.

We initially had no idea of how to extend our notation to cover a zero exponent, but if we wish rules 1, 2 and 3 to remain valid for such an exponent then the definition $b^0 = 1$ is forced on us. We have no choice.

Okay, we have come up with a sensible definition of b^0 by taking $m = n$ in rule 2 and seeing what b^0 must be if rule 2 is to remain valid. To come up with a suitable meaning for negative exponents we can take $n < m$ in rule 2. For example, let's try $n = 2$ and $m = 3$.

Rule 2 gives

$$\begin{aligned}\frac{b^2}{b^3} &= b^{-1} \quad \text{or}\\ \frac{1}{b} &= b^{-1}.\end{aligned}$$

This suggests that we should define b^{-1} to be equal to $\frac{1}{b}$. This definition, too, makes sense for all values of b except $b = 0$.

In a similar way we can see that we should define b^{-n} to mean $\frac{1}{b^n}$, except when $b = 0$, in which case it is undefined. You should convince yourself of this by showing that the requirement that rule 2 remains valid forces on us the definitions

$$\begin{aligned}b^{-2} &= \frac{1}{b^2} \quad \text{and}\\ b^{-3} &= \frac{1}{b^3}.\end{aligned}$$

If n is a positive integer (for example $n = 17$ or $n = 178$) then we define b^{-n} to be equal to $\frac{1}{b^n}$. This definition makes sense for all values of b except $b = 0$, in which case the expression b^{-n} remains undefined.

You should check that, with this definition, rules 1 and 3 also remain valid.

Examples

$3^{12-3\times4} = 3^0 = 1$

$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$

$2^3 \times 1.1^{-4+2} = 2^3 \times 1.1^{-2} = 8 \times 0.826446281 = 6.611570248$

$$\begin{aligned}(x^{-1} + x^{-3})^{-1} &= \frac{1}{x^{-1} + x^{-3}}\\ &= \frac{1}{\frac{1}{x} + \frac{1}{x^3}}\\ &= \frac{1}{\frac{x^2+1}{x^3}}\\ &= \frac{x^3}{x^2+1}\end{aligned}$$

Exercises

Evaluate the following expressions.

11. 5^{-1} **12.** $\frac{3^2}{2^{-3}}$ **13.** 5^{6-9} **14.** $2.2^{4-2\times3}$ **15.** $(6^{-2})^2$

Simplify the following expressions.

16. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}})$

17. $(x^{\frac{1}{4}} - y^{\frac{1}{4}})(x^{\frac{3}{4}} + x^{\frac{1}{2}}y^{\frac{1}{4}} + x^{\frac{1}{4}}y^{\frac{1}{2}} + y^{\frac{3}{4}})$

18. $\frac{x^{\frac{3}{2}} + y^{\frac{1}{2}}x}{x^{\frac{1}{2}} + y^{\frac{1}{2}}}$

19. $\left(\frac{x^{\frac{1}{2}}(x-y)}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})(x^{\frac{1}{2}} + y^{\frac{1}{2}})}\right)^2$

20. $(x^{\frac{1}{2}})^2$

Pause for a moment and look at what has been achieved. We have been able to give a meaning to b^n for all integer values of n, positive, negative, and zero, and we have done it in such a way that all three of the rules above still hold. We can give meaning to expressions like $(\frac{35}{7})^{13}$ and π^{-7}. We have come quite a way, but there are a lot of exponents that we cannot yet handle. For example, what meaning would we give to an expression like $5^{\frac{7}{9}}$? Our next task is to give a suitable meaning to expressions involving fractional powers.

Let us start with $b^{\frac{1}{2}}$. We want to give meaning to this expression in such a way that the rules 1, 2 and 3 remain valid. If rule 2 is to hold then we must have

$$b^{\frac{1}{2}} \times b^{\frac{1}{2}} = b^{\frac{1}{2}+\frac{1}{2}} = b^1 = b.$$

Let's be specific and take $b = 4$. Then, $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4$, so $4^{\frac{1}{2}}$ is equal to a number whose square is 4. There are two numbers whose square is 4. They are 2 and -2. We define $4^{\frac{1}{2}}$ to be the *positive* square root of 4. That is, 2.

In general, $b^{\frac{1}{2}}$ is defined to be the positive square root of b, also written $\sqrt{b}$. So $b^{\frac{1}{2}} = \sqrt{b}$.

Of course, b must be positive if $b^{\frac{1}{2}}$ is to have any meaning for us, because if we take any real number and multiply itself by itself then we get a positive number. (Actually there is a way of giving meaning to the square root of a negative number. This leads to the notion of complex numbers, a beautiful area of mathematics which is beyond the scope of this booklet.)

That takes care of a meaning for $b^{\frac{1}{2}}$ if $b > 0$. Now have a look at $b^{\frac{1}{3}}$. If rule 2 is to remain valid then we must have

$$b^{\frac{1}{3}} \times b^{\frac{1}{3}} \times b^{\frac{1}{3}} = b^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = b^1 = b.$$

As a concrete example take $b = 8$. Then $8^{\frac{1}{3}}$ must be such that $8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8$. There is just one number which when multiplied by itself 3 times gives 8. That number is 2. Thus $8^{\frac{1}{3}} = 2$. For another example take $b = -8$. This time we have no trouble giving a meaning to $(-8)^{\frac{1}{3}}$, even though $-8 < 0$. There is a number which when multipied by itself 3 times gives -8, namely -2, so $(-8)^{\frac{1}{3}} = -2$.

In general if we wish we wish to give meaning to expressions like $b^{\frac{1}{n}}$ in such a way that rule 3 holds then we must have $(b^{\frac{1}{n}})^n = b^1 = b$.

If b is positive, $b^{\frac{1}{n}}$ is defined to be a positive number, the n^{th} root of b. That is, a number whose n^{th} power is equal to b. This number is sometimes written $\sqrt[n]{b}$.

If b is negative we need to look at separately at the cases where n is even and where n is odd.

If n is *even* and b is negative, $b^{\frac{1}{n}}$ cannot be defined, because raising any number to an even power results in a positive number.

If n is *odd* and b is negative, $b^{\frac{1}{n}}$ can be defined. It is a negative number, the n^{th} root of b. For example, $(-27)^{\frac{1}{3}} = -3$ because $(-3) \times (-3) \times (-3) = -27$.

Now we can see how to define $b^{\frac{p}{q}}$ for any number of the form $\frac{p}{q}$, where p and q are integers. Such numbers are called *rational numbers*.

Notice that $\frac{p}{q} = p \times \frac{1}{q}$, so if rule 3 is to hold then

$$b^{\frac{p}{q}} = (b^{\frac{1}{q}})^p = (b^p)^{\frac{1}{q}}.$$

We know how to make sense of $(b^{\frac{1}{q}})^p$ and $(b^p)^{\frac{1}{q}}$, and they turn out to be equal, so this tells us how to make sense of $b^{\frac{p}{q}}$. If we want rules 1, 2 and 3 to hold then we must define $b^{\frac{p}{q}}$ to be either one of $(b^p)^{\frac{1}{q}}$ or $(b^{\frac{1}{q}})^p$.

This definition always makes sense when b is positive, but we must take care when b is negative. If q is even then we may have trouble in making sense of $b^{\frac{p}{q}}$ for negative b.

For example we cannot make sense of $(-3)^{\frac{3}{2}}$. This is because we cannot even make sense of $(-3)^{\frac{1}{2}}$, let alone $((-3)^{\frac{1}{2}})^3$. Trying to take the exponents in the other order does not help us because $(-3)^3 = -27$ and we cannot make sense of $(-27)^{\frac{1}{2}}$.

However it may be that the numerator and denominator of $\frac{p}{q}$ contain common factors which, when cancelled, leave the denominator odd. For example we can make sense of $(-3)^{\frac{4}{6}}$, even though 6 is even, because $\frac{4}{6} = \frac{2}{3}$, and we can make sense of $(-3)^{\frac{2}{3}}$.

A rational number $\frac{p}{q}$ is said to be expressed in its *lowest form* if p and q contain no common factors. If $\frac{p}{q}$, when expressed in its lowest form, has q odd then we can make sense of $b^{\frac{p}{q}}$ even for $b < 0$.

To recapitulate, we define $b^{\frac{p}{q}} = (b^{\frac{1}{q}})^p = (b^p)^{\frac{1}{q}}$. This definition makes sense for all $\frac{p}{q}$ if $b > 0$. If $b < 0$ then this definition makes sense providing that $\frac{p}{q}$ is expressed in its lowest form and q is odd.

So far, if $b > 0$, we have been able to give a suitable meaning to b^x for all rational numbers x. Not every number is a rational number. For example, $\sqrt{2}$ is an *irrational* number: there do not exist integers p and q such that $\sqrt{2} = \frac{p}{q}$. However for $b > 0$ it is possible to extend the definition of b^x to irrational exponents x so that rules 1,2 and 3 remain valid. Thus if $b > 0$ then b^x is defined for all real numbers x and satisfies rules 1, 2 and 3. We will not show how b^x may be defined for irrational numbers x.

Examples

$(\frac{1}{3})^{-1} = \frac{1}{(\frac{1}{3})} = 3$

$(0.2)^{-3} = \frac{1}{(0.2)^3} = \frac{1}{0.008} = 125$

$(-64)^{\frac{2}{3}} = [(-64)^{\frac{1}{3}}]^2 = (-4)^2 = 16$ or,

$(-64)^{\frac{2}{3}} = [(-64)^2]^{\frac{1}{3}} = (4096)^{\frac{1}{3}} = 16$

$16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = 2^3 = 8$

$(-16)^{\frac{3}{4}}$ is not defined.

$5^{\frac{3}{2}} = 5^{1+\frac{1}{2}} = 5 \times 5^{\frac{1}{2}} = 5\sqrt{5}$

Exercises

If the following expressions are not defined then say so. Otherwise evaluate them.

21. $25^{\frac{3}{2}}$ **22.** $(-81)^{\frac{5}{4}}$ **23.** $81^{\frac{5}{4}}$ **24.** $(-27)^{\frac{3}{2}}$ **25.** $(-27)^{\frac{2}{3}}$

1.3 Exponents with Different Bases

From the definition of exponents we know that if n is a positive integer then

$$\begin{aligned} (ab)^n &= \underbrace{(ab) \times (ab) \times \cdots \times (ab)}_{n \text{ factors}} \\ &= \underbrace{a \times a \times \cdots \times a}_{n \text{ factors}} \times \underbrace{b \times b \times \cdots \times b}_{n \text{ factors}} \qquad \text{(switching the order around)} \\ &= a^n b^n. \end{aligned}$$

Just as in section 1.2, we can show that this equation holds true for more general exponents than integers, and we can formulate the following rule:

Rule 4: $(ab)^x = a^x b^x$ whenever both sides of this equation make sense, that is, when each of $(ab)^x$, a^x and b^x make sense.

Again, from the definition of exponents we know that if n is a positive integer then

$$\begin{aligned} \left(\frac{a}{b}\right)^n &= \underbrace{\frac{a}{b} \times \frac{a}{b} \times \cdots \times \frac{a}{b}}_{n \text{ factors}} \qquad (b \neq 0) \\ &= \frac{\overbrace{a \times a \times \cdots \times a}^{n \text{ factors}}}{\underbrace{b \times b \times \cdots \times b}_{n \text{ factors}}} \\ &= \frac{a^n}{b^n} \end{aligned}$$

As in section 1.2, we can show that this equation remains valid if the integer n is replaced by a more general exponent x. We can formulate the following rule:

Rule 5: $(\frac{a}{b})^x = \frac{a^x}{b^x}$ whenever both sides of this equation make sense, that is, whenever $(\frac{a}{b})^x$, a^x and b^x make sense.

An expression of the form $a^x b^y$ cannot generally be simplified, though it can be written in the form $(ab^{\frac{y}{x}})^x$ or $(a^{\frac{x}{y}}b)^y$ if necessary. For example, we cannot really make the expression a^2b^5 any simpler than it is, though we could write it in the form $(ab^{\frac{5}{2}})^2$ or $(a^{\frac{2}{5}}b)^5$.

Examples

$(2 \times 3)^3 = 2^3 \times 3^3 = 8 \times 27 = 216 = 6^3$

$(4x)^{\frac{1}{2}} = 4^{\frac{1}{2}}x^{\frac{1}{2}} = 2x^{\frac{1}{2}} = 2\sqrt{x}$

$(-40)^{\frac{1}{3}} = (-8 \times 5)^{\frac{1}{3}} = (-8)^{\frac{1}{3}} \times (5)^{\frac{1}{3}} = -2 \times \sqrt[3]{5}$

$(\frac{2}{3})^3 = \frac{2^3}{3^3} = \frac{8}{27}$

$(\frac{4}{7})^{-2} = \frac{1}{(\frac{4}{7})^2} = 1 \times \frac{7^2}{4^2} = \frac{49}{16}$

$(-\frac{27}{8})^{-\frac{1}{3}} = (-\frac{8}{27})^{\frac{1}{3}} = \frac{(-8)^{\frac{1}{3}}}{27^{\frac{1}{3}}} = -\frac{2}{3}$

Exercises

Simplify the following expressions.

26. $(x^4y)^{\frac{1}{2}}$ **27.** $6^{\frac{1}{3}} \times 36^{\frac{1}{3}}$ **28.** $x\left(\frac{y^3}{x^4}\right)^{\frac{1}{4}}$ **29.** $(\frac{81}{16})^{\frac{1}{4}}$ **30.** $(a^3b^x)^3$

1.4 Scientific Notation

Scientific notation is a way of expressing any number, especially a very large or a very small number, in a concise and convenient fashion using powers of 10.

For example, consider the numbers 2700000000000000000 and 27000000000000000.

It is plain that they are both quite large numbers. However, written down in this way it is difficult for us to see just how large these numbers are, or to compare their sizes. It is clear that if we did want to compare the size of these numbers we would probably begin by counting the numbers of zeros at the tail of each of these numbers.

The idea behind scientific notation is that any number can be written as a number between 1 and 10 multiplied by a power of 10. For example:

$$\begin{aligned} 45 &= 4.5 \times 10^1 \\ 450 &= 4.5 \times 10^2 \\ 4500 &= 4.5 \times 10^3 \\ 0.45 &= 4.5 \times 10^{-1} \\ 0.045 &= 4.5 \times 10^{-2} \\ 0.0045 &= 4.5 \times 10^{-3}. \end{aligned}$$

Notice that the power to which 10 is raised (the exponent) indicates where to shift the the decimal point. Thus in the expression

$$4500 = 4.5000 \times 10^3$$

the exponent is *positive* and decimal point has been shifted to the *right* by 3 places. Similarly, in

$$0.045 = 4.5 \times 10^{-2}$$

the exponent is *negative* and the decimal point has been shifted to the *left* by 2 places.

To return to the examples given in the beginning of this section, the first number can be written as 2.7×10^{18} and the second as 2.7×10^{16}. Not only is it easier and quicker to write these numbers in scientific notation, but written in this fashion it is clear that the second number is smaller than the first by a factor of $10^2 = 100$.

Remember that in scientific notation it is conventional that the expression is written as a number *between* 1 *and* 10 multiplied by a power of 10. For example we would write 3.7×10^8 rather than 37×10^7 or $.37 \times 10^9$.

Examples

$967 = 9.67 \times 10^2$

$.0000439 = 4.39 \times 10^{-5}$

$8.26 \times 10^{13} = 82600000000000$

$(1.2 \times 10^4) \times (3.7 \times 10^2) = 1.2 \times 3.7 \times 10^{4+2} = 4.44 \times 10^6$

$(3.47 \times 10^{-7}) \times (2.3 \times 10^{23}) = 3.47 \times 2.3 \times 10^{-7+23} = 7.981 \times 10^{-7+23} = 7.981 \times 10^{16}$

$(1.3 \times 10^{-2}) + (7.35 \times 10^{-3}) = .013 + .00735 = .02035 = 2.035 \times 10^{-2}$

$\frac{4.28\times10^{28}}{2.91\times10^{11}} = \frac{4.28}{2.91} \times \frac{10^{28}}{10^{11}} \approx 1.4708 \times 10^{28-11} = 1.4708 \times 10^{17}$

$\frac{3.77\times10^{-8}}{1.15\times10^{-4}} = \frac{3.77}{1.15} \times \frac{10^{-8}}{10^{-4}} \approx 3.278 \times 10^{-8-(-4)} = 3.278 \times 10^{-4}$

Exercises
Write the following numbers in scientific notation.

31. 0.00419 **32.** $3.1 \times 10^2 \times 4.2 \times 10^{-8}$ **33.** $2.7 \times 10^2 + 4.3 \times 10^{-1}$

34. 8327 **35.** $2.1 \times 10^2 \times 8.7 \times 10^{-3}$

1.5 Summary

If $b > 0$ then b^x is defined for all numbers x. If $b < 0$ then b^x is defined for all integers and all numbers of the form $\frac{p}{q}$ where p and q are integers, $\frac{p}{q}$ is expressed in its lowest form and q is odd. The number b is called the *base* and x is called the *power, index* or *exponent*. Exponents have the following properties:

1. If n is a positive integer and b is any real number then $b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ factors}}$.

2. $b^{\frac{1}{n}} = \sqrt[n]{b}$, and if n is even we take this to mean the positive n^{th} root of b.

3. If $b \neq 0$ then $b^0 = 1$. b^0 is undefined for $b = 0$.

4. If p and q are integers then $b^{\frac{p}{q}} = (b^{\frac{1}{q}})^p = (b^p)^{\frac{1}{q}}$.

5. $b^x \times b^y = b^{x+y}$ whenever both sides of this equation are defined.

6. $\dfrac{b^x}{b^y} = b^{x-y}$ whenever both sides of this equation are defined.

7. $b^{-x} = \dfrac{1}{b^x}$ whenever both sides of this equation are defined.

8. $(ab)^x = a^x b^x$ whenever both sides of this equation are defined.

9. $(\dfrac{a}{b})^x = \dfrac{a^x}{b^x}$ whenever both sides of this equation are defined.

Any real number can be written in the form $a \times 10^n$ where a is a number between 0 and 10 and n is an (positive or negative) integer. This is called *scientific notation*.

1.6 Exercises

The following expressions evaluate to quite a 'simple' number. If you leave some of your answers in fractional form you won't need a calculator.

36. $9^{\frac{1}{2}}$ **37.** $16^{\frac{3}{4}}$ **38.** $(\frac{1}{5})^{-1}$ **39.** $(3^{-1})^2$ **40.** $(\frac{5}{2})^{-2}$

41. $(-8)^{\frac{3}{2}}$ **42.** $(\frac{-27}{8})^{\frac{2}{3}}$ **43.** $5^{27}5^{-24}$ **44.** $8^{\frac{1}{2}}2^{\frac{1}{2}}$ **45.** $(-125)^{\frac{2}{3}}$

These look a little complicated but are equivalent to simpler ones. 'Simplify' them. Again, you won't need a calculator.

46. $\dfrac{3^{n+2}}{3^{n-2}}$ **47.** $\sqrt{\dfrac{16}{x^6}}$ **48.** $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$

49. $(x^2 + y^2)^{\frac{1}{2}} - x^2(x^2 + y^2)^{-\frac{1}{2}}$ **50.** $\dfrac{x^{\frac{1}{2}} + x}{x^{\frac{1}{2}}}$ **51.** $(u^{\frac{1}{3}} - v^{\frac{1}{3}})(u^{\frac{2}{3}} + (uv)^{\frac{1}{3}} + v^{\frac{2}{3}})$

Write these numbers in scientific notation.

52. 0.00317 **53.** $2.15 \times 10^7 \times 3.54 \times 10^{-1}$ **54.** $3.47 \times 10^{17} \times 7.4 \times 10^{-3}$

55. $(2.7 \times 10^{65})^{\frac{1}{3}}$ **56.** $5.98 \times 10^6 - 3.7 \times 10^5$ **57.** $\dfrac{3.8 \times 10^{27}}{2.45 \times 10^{-8}}$

Don't bother working these ones out, just decide whether or not the expressions are defined.

58. $(-1.7)^{\frac{1}{8}}$ **59.** $(-3)^{\frac{2}{19}}$ **60.** $(-4.8)^{-\frac{6}{14}}$ **61.** $(\pi)^{\sqrt{2}}$ **62.** $(-\pi)^{-\frac{8}{14}}$

2 Exponential Functions

2.1 The Functions $y = 2^x$ and $y = 2^{-x}$

In the previous section we saw how, if b is a positive number, we can make sense of the expression b^x for all real numbers x. It turns out that functions of the type $y = f(x) = b^x$, where b is a positive number, are of great importance in mathematics and in all branches of the sciences.

To get an indication of how these functions behave we have graphed the function $f(x) = 2^x$ in Figure 1. You should be aware of several important features of this graph.

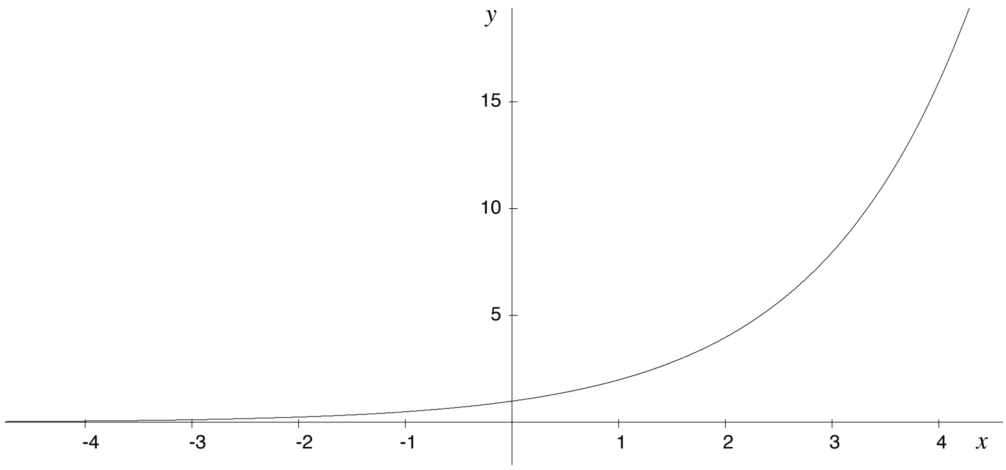

Figure 1: Graph of the function $f(x) = 2^x$

The function $f(x) = 2^x$ is always positive (the graph of the function never cuts the x-axis), although the value of the function gets very close to zero for values of x very large negative (ie a long way to the left along the x-axis). For example, when $x = -5$ we have $2^x = 0.03125$.

The function 2^x increases very rapidly for large values of x. From the rules of exponents discussed in section 1 you should know that $2^{x+1} = 2 \times 2^x$. In words, the value of 2^x doubles if x is increased by 1.

The graph of $y = 2^x$ intercepts the y-axis at $y = 1$. You should expect this because you know from the rules of exponents that $2^0 = 1$.

Figure 2 displays the graph of the function $f(x) = 2^{-x}$. How is the graph of $y = 2^{-x}$ related to the graph of $y = 2^x$? Well, if we set $x = 1$ then $2^{-x} = 2^{-1} = \frac{1}{2}$, which is the value which would have been obtained by setting $x = -1$ in the function $y = 2^x$. In the same way we see that if we set $x = -7$ in the function $y = 2^{-x}$ then we obtain the same value as we would by setting $x = 7$ in the function $y = 2^x$. Proceeding like this we see that *the graph of the function $y = 2^{-x}$ is the reflection in the y-axis of the graph of $y = 2^x$.* Compare Figure 1 with Figure 2.

From the rules of exponents discussed in section 1 it follows that $2^{-x} = (2^{-1})^x = (\frac{1}{2})^x$. The function $y = 2^{-x}$ is the same as the function $y = (\frac{1}{2})^x$, and so

$$2^{-(x+1)} = (\frac{1}{2})^{x+1} = \frac{1}{2} \times (\frac{1}{2})^x = (\frac{1}{2}) \times 2^{-x}.$$

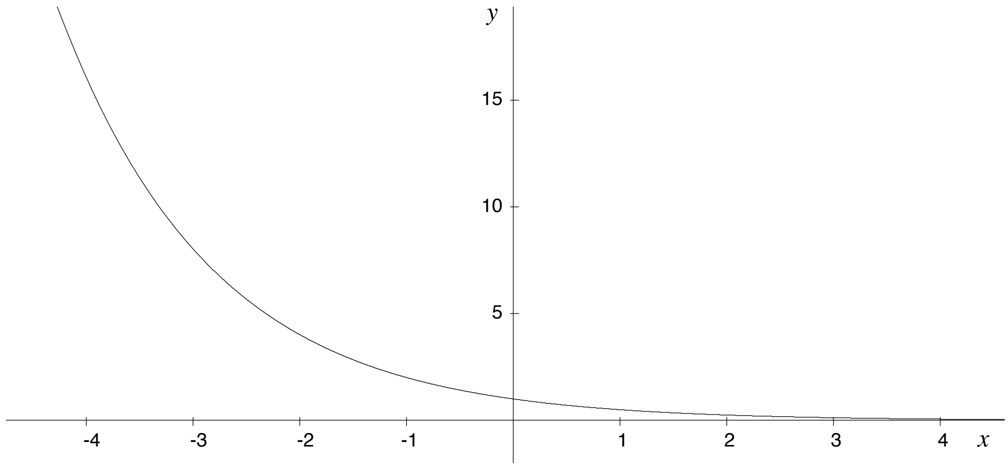

Figure 2: Graph of the function $y = 2^{-x}$

In words, the value of the function $y = 2^{-x}$ is decreased by a factor of $\frac{1}{2}$ if x is increased by 1.

2.2 The functions $y = b^x$ and $y = b^{-x}$

Any function of the form $y = b^x$ where $b > 0$ and $b \neq 1$ behaves like one of the functions $y = 2^x$ or $y = (\frac{1}{2})^x = 2^{-x}$.

If $b > 1$ then the function $y = b^x$ is increasing and behaves like $y = 2^x$.

If $b < 1$ then the function is decreasing and behaves like $y = (\frac{1}{2})^x = 2^{-x}$.

If $b = 1$ then $y = 1^x = 1$ for all x. Notice that regardless of the value of b, providing always that $b > 0$, the function $y = b^x$ intercepts the y-axis at $y = 1$. This is because $b^0 = 1$ for all numbers b. Figure 3 shows the graphs of the functions $y = b^x$ for various values of b.

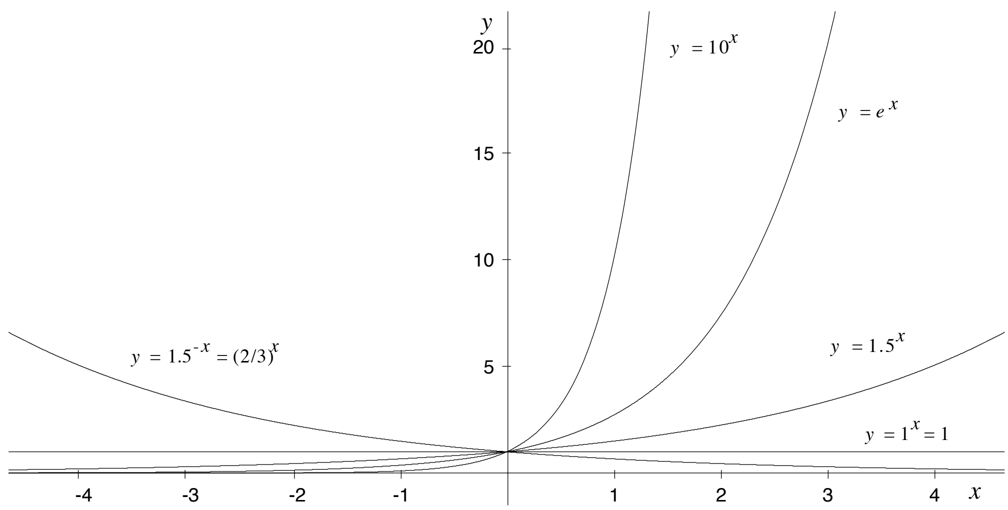

Figure 3: Graphs of $y = b^x$ for various values of b.

Exercises

1. Make a careful sketch of the graphs of the functions $y = 2.5^x$ and $y = 5^{-x}$. Indicate where (if at all) these functions intercept the axes.

2. Which of the following functions are increasing and which are decreasing? You should be able to decide without graphing the functions or substituting any values, though you may do so if you wish.

a. $f(x) = 2.7^x$ **b.** $f(x) = (\frac{1}{2.7})^{-x}$ **c.** $f(x) = 3^{-x}$ **d.** $f(x) = 0.22^x$

2.3 The Functions $y = e^x$ and $y = e^{-x}$

There is a number called e which has a special importance in mathematics. Like the number π, the number e is an *irrational* number (see section 1.2), which is equivalent to saying that it has a non-terminating, non-repeating decimal representation. In other words we can never write down exactly what e is. To 5 decimal places it is equal to 2.71828, but this is just an approximation of the correct value. Unless you really need to write down an approximate value for e it is more convenient and accurate to leave the symbol e in expressions involving this number. For example, it is preferable to write $2e$ rather than 2×2.71828 or 5.43656.

In mathematics the functions e^x and e^{-x} are particularly important. Because of this we have graphed them in figure 4. You can see how similar these functions are to the other exponential functions. The reasons for their importance are discussed briefly in section 5.5.

The function $y = e^x$ is often referred to as *the* exponential function, and is even given another special symbol, exp, so that $\exp(x) = e^x$ and $\exp(-x) = e^{-x}$.

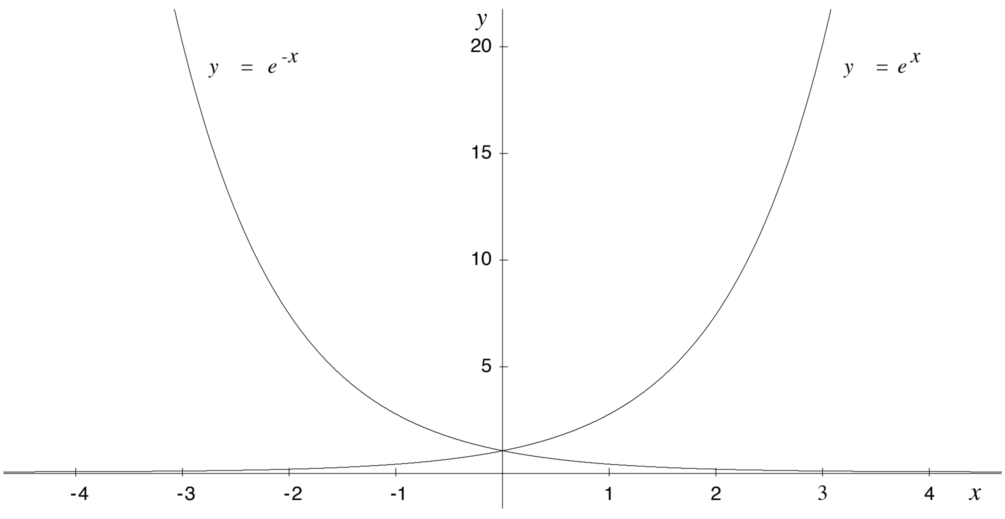

Figure 4: Graphs of e^x and e^{-x}.

2.4 Summary

Functions of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$ are called *exponential functions.*

If $b < 1$ then b^x is a decreasing function, and if $b > 1$ then b^x is an increasing function.

The function b^{-x} is equal to the function $(\frac{1}{b})^x$.

The number $e \approx 2.71828$ and the functions e^x and e^{-x} are of special importance in mathematics. The function e^x is often given the special name exp, so that $\exp(x) = e^x$ and $\exp(-x) = e^{-x}$.

2.5 Exercises

3. Which of the following functions are increasing and which are decreasing? If you have understood this section fully you will be able to answer this question without graphing the functions or substituting any values.

a. $f(x) = (\frac{5}{3})^x$ **b.** $f(x) = (\frac{5}{3})^{-x}$ **c.** $f(x) = (\frac{3}{5})^{-x}$ **d.** $f(x) = (\frac{3}{5})^x$

4. Sketch the graphs of the functions $f(x) = 3^x$ and $f(x) = 3^{-x}$. On the same diagrams mark in roughly the graphs of $f(x) = 2.9^x$ and 2.9^{-x}.

5. It is true that $e^{1.09861} \approx 3$. Try it for yourself on a calculator if you have one. How do you think the functions $y = 3^x$ and $y = e^{1.09861x}$ compare? Why? If you cannot solve this otherwise, you might like to try substituting in a few numbers for x in both of the functions and comparing the values.

3 Logarithms

3.1 Introduction

Taking logarithms is the reverse of taking exponents, so you must have a good grasp on exponents before you can hope to understand logarithms properly. Review the material in the first two sections of this booklet if necessary.

We begin the study of logarithms with a look at logarithms to base 10. It is important that you realise from the beginning that, as far as logarithms are concerned, there is nothing special about the number 10. Indeed, the most natural logarithms are logarithms to base e, and they are introduced in section 3.4. Logarithms to base 10 are in common use only because we use a decimal system of counting, and this is probably a result of the fact that humans have ten fingers. We have begun with logarithms to base 10 only to be definite, and we could just as easily have started with logarithms to any other convenient base.

3.2 Logarithms to Base 10 (Common Logarithms)

We will begin by considering the function $y = 10^x$, graphed in Figure 5. As we know

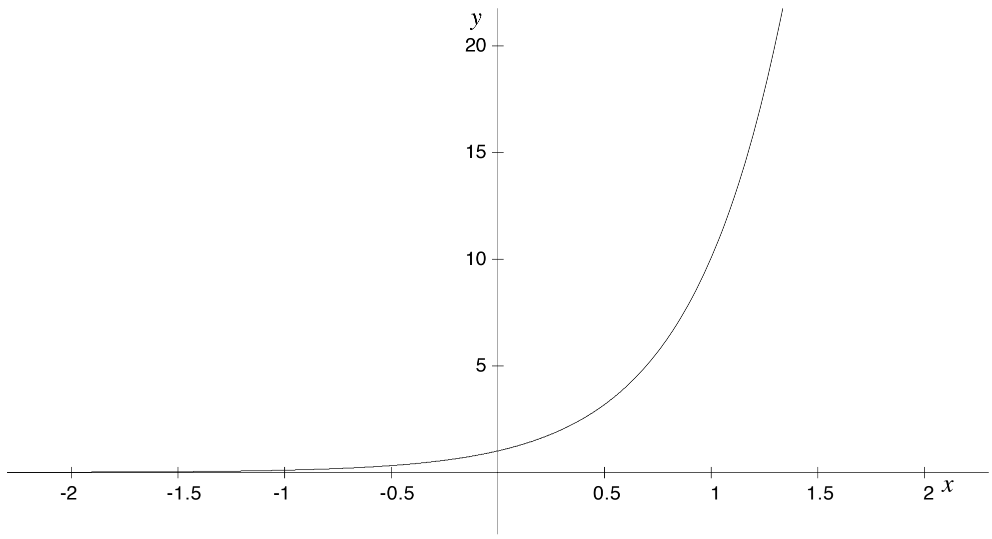

Figure 5: Graph of $f(x) = 10^x$

from the discussion in section 1, given any number x, we can raise 10 to the power of x to obtain another number which we write as 10^x. What of the reverse procedure? Suppose we begin with a number and we wish to find the power to which 10 must be raised to obtain that number.

For example, suppose we begin with the number 7 and we wish to find the power to which 10 must be raised to obtain 7. This number is called the *logarithm to the base* 10 *of* 7 and is written $\log_{10} 7$. Similarly, $\log_{10} 15$ is equal to the power to which 10 must be raised to obtain 15.

For a general number x, $\log_{10} x$ is equal to that power to which 10 must be raised to obtain the number x.

When we see an expresion like $\log_{10} 29$ we can think of it as a sort of a question. The question we have in mind is this: to what power must we raise 10 to get 29? Or, $10^? = 29$. The answer to this question is a number, and we call that number $\log_{10} 29$.

The definition of the logarithm to base 10 is the basis on which the remainder of this section rests, and it is extremely important that you understand it properly.

Again: $\log_{10} x$ *is equal to that power to which* 10 *must be raised to obtain the number* x.

As an example, let's calculate $\log_{10} 10^3$. According to the definition, $\log_{10} 10^3$ is equal to that power to which 10 must be raised to obtain 10^3. To what power must we raise 10 to obtain 10^3? Or, $10^? = 10^3$. Surely the answer is 3. Notice that $10^3 = 1000$, so we have worked out $\log_{10} 1000$, and without using a calculator! We have been able to work this out because we have understood the *meaning* of the logarithm of a number. We will need to use a calculator to work out the logarithms of most numbers, but it is very important that we understand what it is that the calculator is working out for us when we push the buttons.

Without a calculator we can work out the logarithms of many numbers.

Examples:

$$\begin{aligned}\log_{10} 100 &= \log_{10} 10^2 = 2\\ \log_{10} 0.1 &= \log_{10} 10^{-1} = -1\\ \log_{10} 10\sqrt{10} &= \log_{10} 10^{1.5} = 1.5\end{aligned}$$

Exercise

1. By expressing these numbers as powers of 10, and without using a calculator, calculate the logarithms to base 10 of the following numbers.

a. 10000	**b.** $\frac{1}{100}$	**c.** 0.001	**d.** $10^{12.3}$
e. $\sqrt{10}$	**f.** $10\sqrt[4]{10}$	**g.** $(\frac{1}{1000})\sqrt[3]{10}$	**h.** $\frac{1}{0.001}$

Can we take the logarithm of any number? In other words, given any number x can we find a power to which 10 may be raised to obtain the number x?

Look at the graph of $y = 10^x$ in Figure 5. We see that 10^x is *never* negative and indeed never even takes the value 0. *There is no power to which we may raise* 10 *to obtain a number less than or equal to* 0. This means that we cannot take the logarithm of a number less than or equal to zero. We say that $\log_{10} x$ is *undefined* for $x \le 0$.

The graph of 10^x gives us another important piece of information. If $x > 0$ then there is only one power to which we may raise 10 to get x. Our definition of $\log_{10} x$ is unambiguous.

The graph of $y = \log_{10} x$ is shown in Figure 6. You should pay attention to several important features of this graph.

The graph intercepts the x-axis at $x = 1$. In other words, $\log_{10} 1 = 0$. You should expect this because you should know from section 1 that $10^0 = 1$.

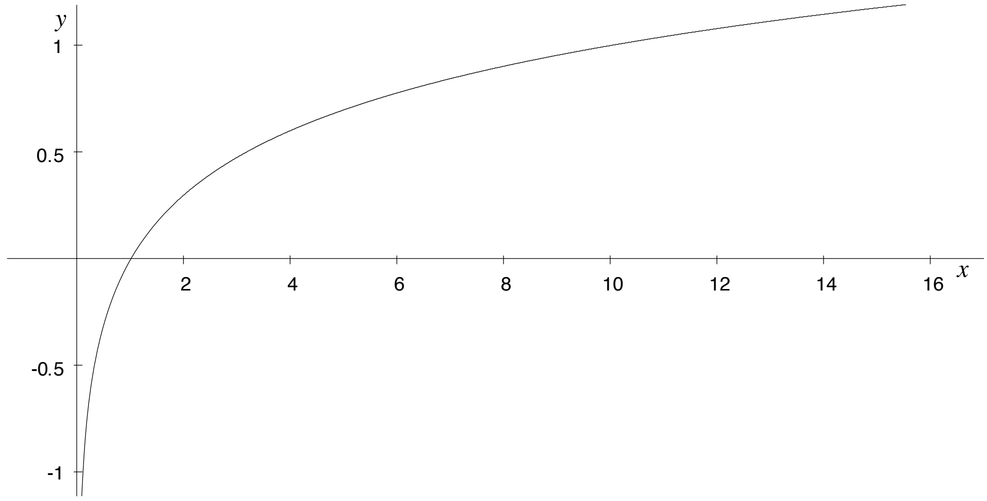

Figure 6: Graph of $f(x) = \log_{10} x$

The graph does not extend to the left of the y-axis, and in fact never even intercepts the y-axis. We have already commented on the fact that the logarithm of a number less than or equal to zero is not defined.

The function $y = \log_{10} x$ gets as large as we like as x gets large. By this we mean that we can make $\log_{10} x$ as large as we choose by choosing x to be sufficiently large. The graph does not stay below a certain height as x gets large (it does not have a horizontal asymptote). However the function $y = \log_{10} x$ increases very slowly as x increases.

The fact that we bother to specify the base as being 10 suggests that we can take logarithms to other bases. We can, and we shall say more about this later, but for now let us stick with base 10.

You should be aware that many writers may not mention the base of the logarithms they are referring to if it is obvious from the context what that base is, or if it does not matter which base is used. They may just write 'the logarithm of x' or $\log x$.

Because logarithms to base 10 have been used so often they are called *common logarithms.* If you have a calculator it probably has a Log button on it. You could use it to find, for example, $\log_{10} 7$ and $\log_{10} 0.01$.

From the examples above you should be able to see that if we express a number as a power of 10 then we can read off the logarithm to base 10 of that number from the power. Let's try to make this precise. Suppose that x is any real number. What is $\log_{10} 10^x$? Well, $\log_{10} 10^x$ is that power to which 10 must be raised to obtain the number 10^x. To what power must we raise 10 to obtain the number 10^x? Or, to put this question another way, $10^? = 10^x$. The answer must be x. Thus $\log_{10} 10^x = x$. This is our first rule of logarithms.

Rule A: For any real number x, $\log_{10} 10^x = x$.

Examples

$$\begin{aligned}\log_{10} 10^{3.7} &= 3.7\\ \log_{10} 0.0001 &= \log_{10} 10^{-4} = -4\\ \log_{10} 10^4 \sqrt[5]{10^3} &= \log_{10} 10^4 \times (10^3)^{\frac{1}{5}} = \log_{10} 10^{4+\frac{3}{5}} = \frac{23}{5}\end{aligned}$$

Rule A tells us what happens if we first raise 10 to the power x and then take the logarithm to base 10 of the result. We end up with what we started with. What happens if we do things in the reverse order? Consider the number $\log_{10} 7$. If you have a calculator with a Log button on it you can see that this number is approximately 0.8451. Now suppose we raise 10 to the power $\log_{10} 7$. What do you think the result is? In symbols, what is $10^{\log_{10} 7}$? Well, remember that $\log_{10} 7$ is equal to that power to which 10 must be raised to give the number 7. So if we raise 10 to that power then we must get 7. The same reasoning applies to show that if $x > 0$ then $10^{\log_{10} x} = x$. The number $\log_{10} x$ is that power to which 10 must be raised to obtain x. So if we raise 10 to this power we must get x. We will write this down as the second of our rules of logarithms.

Rule B: For any real number $x > 0$, $10^{\log_{10} x} = x$.

Examples

$$\begin{aligned}10^{\log_{10} \pi} &= \pi\\ 10^{\log_{10}(x^2+y^2)} &= x^2 + y^2\\ 10^{\log_{10} 10^{3x^3}} &= 10^{3x^3}\end{aligned}$$

Exercises

2. Simplify the following expressions.

a. $10^{\log_{10} 37.23}$ **b.** $\log_{10} 10^{x^2 y}$ **c.** $10^{\log_{10}(10^x)}$ **d.** $\log_{10} \sqrt{10}^{\frac{x}{2y}}$

e. $10^{10^{\log_{10} x}}$ **f.** $\log_{10} 10^{\frac{x+y}{z}}$ **g.** $10^{\log_{10}(\frac{3xy}{z})}$ **h.** $\log_{10} 10^{10^{2x}}$

Rules A and B express the fact that the functions $y = 10^x$ and $y = \log_{10} x$ are *inverse functions* of one another. If you have not come across the concept of inverse functions before then do not worry about what this means. If you have, then you will probably remember that the graph of an inverse function is obtained by reflecting the graph of the original function in the line $y = x$, that is the line which runs in the north-east and south-west direction. Take another look at Figures 5 and 6.

We can use the rules of exponents discussed in section 1 to work out more rules for logarithms.

If x and y are numbers greater than zero then, by rule B, $x = 10^{\log_{10} x}$ and $y = 10^{\log_{10} y}$, so

$$\begin{aligned}xy &= 10^{\log_{10} x} \times 10^{\log_{10} y}\\ &= 10^{\log_{10} x + \log_{10} y} \qquad \text{(by the rules for exponents).}\end{aligned}$$

This equation tell us that if we raise 10 to the power $\log_{10} x + \log_{10} y$ then we get the number xy. In other words it tells us that $\log_{10} x + \log_{10} y$ is the answer to the question $10^{?} = xy$. But the answer to this question is also $\log_{10} xy$. Thus $\log_{10} xy = \log_{10} x + \log_{10} y$. This we will call our third rule of logarithms.

Rule C: For any real numbers $x > 0$ and $y > 0$, $\log_{10} xy = \log_{10} x + \log_{10} y$.

So much for multiplication. What of division? If $x > 0$ and $y > 0$ then

$$\begin{aligned} \frac{x}{y} &= \frac{10^{\log_{10} x}}{10^{\log_{10} y}} && \text{(by rule B)} \\ &= 10^{\log_{10} x - \log_{10} y} && \text{(by the rules for exponents).} \end{aligned}$$

This equation tells us that if we raise 10 to the power $\log_{10} x - \log_{10} y$ then we get the number $\frac{x}{y}$. In other words, $\log_{10} \frac{x}{y} = \log_{10} x - \log_{10} y$. This is our fourth rule of logarithms.

Rule D: For any real numbers $x > 0$ and $y > 0$, $\log_{10}(\frac{x}{y}) = \log_{10} x - \log_{10} y$.

If x is a number, $x > 0$, and n is any number at all then:

$$\begin{aligned} x^n &= (10^{\log_{10} x})^n && \text{(by rule B)} \\ &= 10^{n \times \log_{10} x} && \text{(by the rules for exponents).} \end{aligned}$$

This equation tells us that if we raise 10 to the power $n \log_{10} x$ then we get the number x^n. In other words, $\log_{10} x^n = n \log_{10} x$. This is our fifth rule of logarithms.

Rule E: For real numbers x and n, with $x > 0$, $\log_{10} x^n = n \log_{10} x$

Examples

$$\begin{aligned} \log_{10} \frac{xy}{z} &= \log_{10} x + \log_{10} y - \log_{10} z \\ \log_{10} x^3 y^{-2} &= 3 \log_{10} x - 2 \log_{10} y \\ 2 \log_{10} y - 4 \log_{10}(x^2 - z^3) &= \log_{10} \frac{y^2}{(x^2 - z^3)^4} \end{aligned}$$

Exercises

3. Rewrite the following expressions so that they involve just one logarithm.

a. $\log_{10} x^3 - 2.5 \log_{10} y$

b. $\log_{10} 6 + \log_{10} x^{-2}$

c. $5 \log_{10} 3x - 4 \log_{10}(xy + z^2)$

d. $2 \log_{10} xy + 3 \log_{10}(z^2 - y^2)$

e. $\log_{10}(x + y) - 3 \log_{10} 4$

f. $\log_{10} xy - 1.7 \log_{10} y^2$

3.3 Logarithms to Base b

As we mentioned above, we can take logarithms to other bases. If b is a real number, $b > 1$, and if x is a real number, $x > 0$, then we define the logarithm to base b of x to be that power to which b must be raised to obtain the number x.

You may also think of $\log_b x$ as the answer to the question $b^? = x$. You should notice that if $b = 10$ then this definition agrees with the one given earlier for $\log_{10} x$.

Again: *the logarithm to base b of a number* $x > 0$ (written $\log_b x$) *is that power to which b must be raised to obtain the number x.*

Examples:

$$\begin{aligned}\log_5 125 &= \log_5 5^3 = 3\\ \log_{16} 2 &= \log_{16} 16^{\frac{1}{4}} = \frac{1}{4}\\ \log_7 \frac{1}{49} &= \log_7 7^{-2} = -2\end{aligned}$$

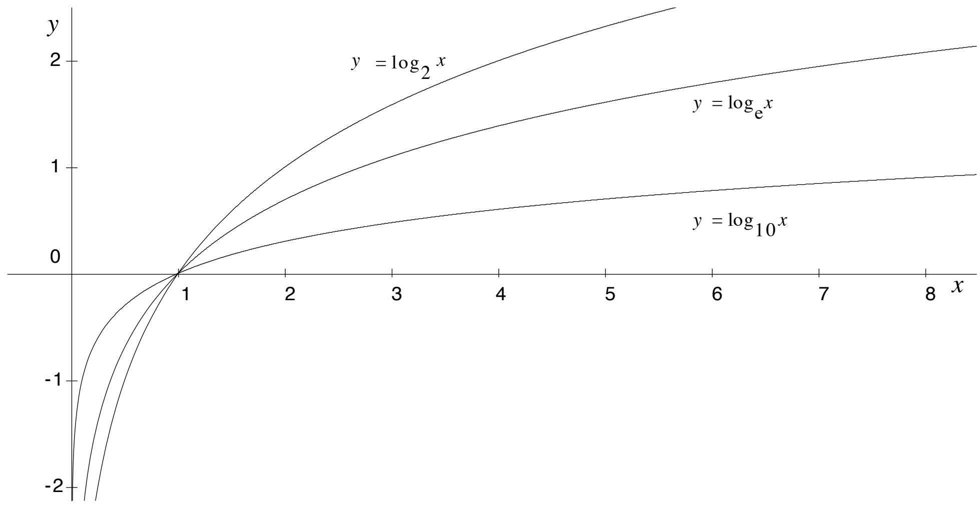

Figure 7: Graph of $f(x) = \log_b x$ for various values of b.

We have required the base of our logarithms, b, to be greater than 1. In fact we can take logarithms to any base b provided $b > 0$ and $b \neq 1$. It is more usual though to use $b > 1$, and in this booklet we will always use a base $b > 1$.

Figure 7 shows graphs of the functions $y = \log_b x$ for various values of b. As you can see from these graphs, the logarithm functions behave in a similar fashion for different bases b, providing $b > 1$.

All of what we said earlier remains true for $\log_b x$ if 10 is replaced by b. In particular the five rules of logarithms remain true. Let us restate these to be applicable to $\log_b x$.

For a real number $b > 1$:

Rule 1: For any real number x, $\log_b b^x = x$

Rule 2: For any real number $x > 0$, $b^{\log_b x} = x$

Rule 3: For any real numbers $x > 0$ and $y > 0$, $\log_b xy = \log_b x + \log_b y$

Rule 4: For any real numbers $x > 0$ and $y > 0$, $\log_b \frac{x}{y} = \log_b x - \log_b y$

Rule 5: For real numbers x and n, with $x > 0$, $\log_b x^n = n \log_b x$

Exercises

4. Simplify these expressions.

a. $\log_2 2^x 2^{x+y}$ **b.** $5^{\log_5 \frac{x+y}{3}}$ **c.** $\log_7 49^{uv}$ **d.** $3^{\log_9 \frac{u}{3w}}$

5. Rewrite the following expressions so that they involve only one logarithm.

a. $2\log_3(x+y) - 3\log_3(xy) + \log_3 x^2$ **b.** $\log_6 xy - 4\log_6(x+y)$

c. $4\log_{17} xy^2 + \log_{17}(x^2+y^2) - 2.5\log_{17} x$

Now that we have shown how to define logarithms to any base $b > 1$, let us see how these logarithms are related to each other. We will consider logarithms to two bases $a > 1$ and $b > 1$. By rule 2,

$$x = a^{\log_a x}.$$

Taking logarithms to base b of both sides of this equation yields

$$\begin{aligned} \log_b x &= \log_b(a^{\log_a x}) \\ &= \log_a x \times \log_b a \qquad \text{(by rule 5).} \end{aligned}$$

This, our sixth rule of logarithms, tells us how logarithms to different bases are related.

Rule 6: For numbers $x > 0$, $a > 1$ and $b > 1$, $\log_b x = \log_b a \times \log_a x$.

From this rule we see that $\log_b a \times \log_a b = \log_b b = 1$, and so

$$\log_b a = \frac{1}{\log_a b}.$$

This fact enables us to calculate the logarithm of a number to any base from a calculator which calculates logarithms to one base only.

Example: If your calculator only has logarithms to base 10 on it, how can you find $\log_7 9$?

Solution: By rule 6,

$$\begin{aligned} \log_7 9 &= \log_7 10 \times \log_{10} 9 \\ &= \frac{1}{\log_{10} 7} \times \log_{10} 9 \end{aligned}$$

and the last expression can be evaluated by any calculator which can evaluate logarithms to base 10.

Exercises

6. Using a calculator, find the following logarithms.

a. $\log_3 17$ **b.** $\log_5 2$ **c.** $\log_{22} 14$ **d.** $\log_4 8$

3.4 Logarithms to Base e (Natural Logarithms)

Logarithms to the base 10 are commonly used, because we use a decimal number system and not a base 8 system, or a base 2 system. If humans were born with 3 toes (or if sloths could count) then logarithms to base 3 might be in common use. Apart from the fact that we use a decimal number system, there is no reason for us to prefer logarithms to base 10 over logarithms to any other base. Indeed, we mentioned in section 2 that the function $y = e^x$ is a very important function in mathematics, and it is therefore reasonable to expect that logarithms to base e will also assume special importance.

They do, and are given the name 'Natural Logarithms' or 'Napierian Logarithms'. They are even given a special symbol, ln, so that $\ln x = \log_e x$. One of the graphs in Figure 7 is a plot of the function $y = \log_e x = \ln x$. Notice that the function $y = \ln x$ behaves in a similar fashion to the function $y = \log_{10} x$. This comes as no surprise to us since we have already seen in section 2 that the functions e^x and 10^x are very similar to each other.

3.5 Exponential Functions Revisited

In section 2 we saw how much the exponential functions resemble each other. If $b > 1$ then the exponential function b^x looks very much like any of the other exponential functions with base greater than 1, and if $b < 1$ then b^x looks a lot like any of the exponential functions with base less than one. We will now be able to see more clearly what is going on here.

Consider the function $y = 2^x$. Now $2 = e^{\log_e 2}$, so we can write

$$2^x = (e^{\log_e 2})^x = e^{x \log_e 2}.$$

We have been able to write the function 2^x as a function involving the base e, though the exponent is now not simply x, but x multiplied by some fixed number, namely $\log_e 2$.

Similarly, we could write

$$\begin{aligned} 5^x &= e^{x \log_e 5} \\ 19^{-x} &= e^{-x \log_e 19} \\ 7^{-x} &= 4^{-x \log_4 7} \\ e^x &= 13^{x \log_{13} e} \end{aligned}$$

We can write all exponential functions in the form $y = e^{kx}$, where k is some constant which may be negative.

Exercises

7. Write each of the following functions in the form $y = e^{kx}$ for a suitable constant k.

a. $y = 10^x$ **b.** $y = 7.5^{-x}$ **c.** $y = 4^{-x}$ **d.** $y = (\frac{1}{4})^x$

3.6 Summary

For any real number $b > 1$ and any $x > 0$, $\log_b x$ is equal to that number to which b must be raised to obtain the number x. One can think of $\log_b x$ as the answer to the question $b^? = x$. The number $\log_b x$ is called the *logarithm to base b of x.*

The function $\log_b x$ satisfies the following rules:

Rule 1: For any real number x, $\log_b b^x = x$

Rule 2: For any real number $x > 0$, $b^{\log_b x} = x$

Rule 3: For any real numbers $x > 0$ and $y > 0$, $\log_b xy = \log_b x + \log_b y$

Rule 4: For any real numbers $x > 0$ and $y > 0$, $\log_b \frac{x}{y} = \log_b x - \log_b y$

Rule 5: For real numbers x and n, with $x > 0$, $\log_b x^n = n \log_b x$

Rule 6: For numbers $x > 0$, $a > 1$ and $b > 1$, $\log_b x = \log_b a \times \log_a x$.

Logarithms to base 10 are in common use and for this reason they are called Common Logarithms.

Logarithms to base e are of special importance. They are often called natural logarithms or Napierian logarithms, and the symbol $\ln x$ is often used for them. Thus $\ln x = \log_e x$.

Any exponential function may be written in the form e^{kx}, where the constant k may be negative.

3.7 Exercises

Without using a calculator, find the following numbers.

8. $\log_{10} 10^{-19}$ **9.** $\log_e e\sqrt[5]{e}$ **10.** $\log_2 16$

11. $\log_{17} \frac{17^3}{\sqrt{17}}$ **12.** $\ln \frac{e^2}{e^{21}}$ **13.** $\frac{\ln e^7}{\log_{11} 121}$

14. $5^{\log_5 32.7}$ **15.** $e^{\ln \frac{9}{2}}$ **16.** $e^{\ln \sqrt[3]{27}}$

Rewrite the following expressions using the rules of logarithms, and simplify where possible.

17. $\log_{10} \frac{100x^2}{9y}$ **18.** $\ln \frac{xy^{-3}}{e^{1.37}}$ **19.** $\log_4 \frac{4^{-1.3}z^7}{x^2y^3}$

20. $\log_3 \frac{x^3y^2}{27z^{\frac{1}{2}}}$ **21.** $\ln(e^{-2.4}x^6)$ **22.** $\log_5 \frac{125x^3}{0.2y^2}$

Using the rules of logarithms, rewrite the following expressions so that just one logarithm appears in each.

23. $3\log_2 x + \log_2 30 + \log_2 y - \log_2 w$ **24.** $2\ln x - \ln y + a\ln w$

25. $12(\ln x + \ln y)$ **26.** $\log_3 e \times \ln 81 + \log_3 5 \times \log_5 w$

27. $\log_7 10 \times \log_{10} x^2 - \log_7 49x$ **28.** $\log_{10} 0.1 \times \log_6 x - 2\log_6 y + \log_6 4 \times \log_4 e$

Given that $\log_e 5 \approx 1.6094$, and $\log_e 7 \approx 1.9459$, find the following numbers without using a calculator except to perform multiplication or division.

29. $\log_5 e$ **30.** $\log_5 7$ **31.** $\log_5 7^2$ **32.** $\log_{49} 5$ **33.** $\log_{49} 25$ **34.** $\log_e 25$

4 Solutions to Exercises

4.1 Solutions to Exercises from Section 1

1. $3^4 = 3 \times 3 \times 3 \times 3 = 81$

2. $4^2 \times 3^3 = 4 \times 4 \times 3 \times 3 \times 3 = 16 \times 27 = 432$

3. $\frac{1.5^7}{1.5^4} = 1.5^{7-4} = 3.375$

4. $(2.7^2)^3 = 2.7^{2\times 3} = 387.420489$

5. $(3 + 2.2^3)^4 = 13.648^4 = 34695.732$

6. $\frac{2^{2x+y}}{2^x} = 2^{2x+y-x} = 2^{x+y} = 2^x 2^y$

7. $\frac{x^7 x^{9.5}}{x^{1.2}} = x^{7+9.5-1.2} = x^{15.3}$

8. $4^{x+3y} = 4^x 4^{3y}$

9. $(3^x)^z = 3^{xz}$

10. $\frac{3^{x+2z}}{3^{\frac{z}{4}}} = 3^{x+2z-\frac{z}{4}} = 3^{x+\frac{7z}{4}}$

11. $5^{-1} = \frac{1}{5}$

12. $\frac{3^2}{2^{-3}} = \frac{9}{\frac{1}{8}} = 72$

13. $5^{6-9} = 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

14. $2.2^{4-2\times 3} = 2.2^{-2} = \frac{1}{2.2^2} \approx .2066116$

15. $(6^{-2})^2 = 6^{-2\times 2} = \frac{1}{6^4} = .0007716049382716$

16.

$$\begin{aligned} (x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}}) &= x^{\frac{1}{2}}x^{\frac{1}{2}} - x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{1}{2}}x^{\frac{1}{2}} - y^{\frac{1}{2}}y^{\frac{1}{2}} \\ &= x - y \end{aligned}$$

17.

$$\begin{aligned} (x^{\frac{1}{4}} - y^{\frac{1}{4}})(x^{\frac{3}{4}} + x^{\frac{1}{2}}y^{\frac{1}{4}} + x^{\frac{1}{4}}y^{\frac{1}{2}} + y^{\frac{3}{4}}) &= x^{\frac{1}{4}}x^{\frac{3}{4}} + x^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}} + x^{\frac{1}{4}}x^{\frac{1}{4}}y^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{3}{4}} \\ &\quad - y^{\frac{1}{4}}x^{\frac{3}{4}} - y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}} - y^{\frac{1}{4}}x^{\frac{1}{4}}y^{\frac{1}{2}} - y^{\frac{1}{4}}y^{\frac{3}{4}} \\ &= x + x^{\frac{3}{4}}y^{\frac{1}{4}} + x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{3}{4}} \\ &\quad - x^{\frac{3}{4}}y^{\frac{1}{4}} - x^{\frac{1}{2}}y^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{3}{4}} - y \\ &= x - y \end{aligned}$$

18.

$$\begin{aligned} \frac{x^{\frac{3}{2}} + y^{\frac{1}{2}}x}{x^{\frac{1}{2}} + y^{\frac{1}{2}}} &= \frac{x(x^{\frac{1}{2}} + y^{\frac{1}{2}})}{x^{\frac{1}{2}} + y^{\frac{1}{2}}} \\ &= x \end{aligned}$$

19.

$$\begin{aligned}\left(\frac{x^{\frac{1}{2}}(x-y)}{(x^{\frac{1}{2}}-y^{\frac{1}{2}})(x^{\frac{1}{2}}+y^{\frac{1}{2}})}\right)^2 &= \left(\frac{x^{\frac{1}{2}}(x-y)}{x^{\frac{1}{2}}x^{\frac{1}{2}}+x^{\frac{1}{2}}y^{\frac{1}{2}}-x^{\frac{1}{2}}y^{\frac{1}{2}}-y^{\frac{1}{2}}y^{\frac{1}{2}}}\right)^2 \\ &= \left(\frac{x^{\frac{1}{2}}(x-y)}{x-y}\right)^2 \\ &= \left(x^{\frac{1}{2}}\right)^2 \\ &= x\end{aligned}$$

20. $(x^{\frac{1}{2}})^2 = x^{\frac{1}{2}\times 2} = x$

21. $25^{\frac{3}{2}} = (25^{\frac{1}{2}})^3 = 5^3 = 125$

22. $(-81)^{\frac{5}{4}}$ is not defined.

23. $81^{\frac{5}{4}} = (81^{\frac{1}{4}})^5 = 3^5 = 243$

24. $(-27)^{\frac{3}{2}}$ is not defined.

25. $(-27)^{\frac{2}{3}} = ((-27)^{\frac{1}{3}})^2 = (-3)^2 = 9$

26. $(x^4y)^{\frac{1}{2}} = x^{4\times\frac{1}{2}}y^{\frac{1}{2}} = x^2y^{\frac{1}{2}}$

27. $6^{\frac{1}{3}} \times 36^{\frac{1}{3}} = (6\times 36)^{\frac{1}{3}} = 6$

28. $x\left(\frac{y^3}{x^4}\right)^{\frac{1}{4}} = xy^{\frac{3}{4}}x^{-1} = y^{\frac{3}{4}}$

29. $(\frac{81}{16})^{\frac{1}{4}} = \frac{81^{\frac{1}{4}}}{16^{\frac{1}{4}}} = \frac{3}{2}$

30. $(a^3b^x)^3 = a^{3\times 3}b^{3\times x} = a^9b^{3x}$

31. $0.00419 = 4.19\times 10^{-3}$

32. $3.1\times 10^2\times 4.2\times 10^{-8} = 3.1\times 4.2\times 10^{2-8} = 13.02\times 10^{-6} = 1.302\times 10^{-5}$

33. $2.7\times 10^2 + 4.3\times 10^{-1} = 270 + 0.43 = 270.43 = 2.7043\times 10^2$

34. $8327 = 8.327\times 10^3$

35. $2.1\times 10^2\times 8.7\times 10^{-3} = 2.1\times 8.7\times 10^{2-3} = 18.27\times 10^{-1} = 1.827$

36. $9^{\frac{1}{2}} = \sqrt{9} = 3$

37. $16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = 2^3 = 8$

38. $(\frac{1}{5})^{-1} = \frac{1}{\frac{1}{5}} = 5$

39. $(3^{-1})^2 = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

40. $(\frac{5}{2})^{-2} = (\frac{2}{5})^2 = \frac{4}{25}$

41. $(-8)^{\frac{3}{2}}$ is not defined.

42. $(\frac{-27}{8})^{\frac{2}{3}} = ((\frac{-27}{8})^{\frac{1}{3}})^2 = (\frac{-3}{2})^2 = \frac{9}{4}$

43. $5^{27}5^{-24} = 5^{27-24} = 5^3 = 125$

44. $8^{\frac{1}{2}}2^{\frac{1}{2}} = (8 \times 2)^{\frac{1}{2}} = 16^{\frac{1}{2}} = 4$

45. $(-125)^{\frac{2}{3}} = ((-125)^{\frac{1}{3}})^2 = (-5)^2 = 25$

46. $\frac{3^{n+2}}{3^{n-2}} = 3^{n+2-(n-2)} = 3^4 = 81$

47. $\sqrt{(\frac{16}{x^6})} = (\frac{16}{x^6})^{\frac{1}{2}} = \frac{16^{\frac{1}{2}}}{x^{6\times\frac{1}{2}}} = \frac{4}{x^3}$

48. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 = (a^{\frac{1}{2}})^2 + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + (b^{\frac{1}{2}})^2 = a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b$

49.

$$\begin{aligned}
(x^2+y^2)^{\frac{1}{2}} - x^2(x^2+y^2)^{-\frac{1}{2}} &= (x^2+y^2)^{\frac{1}{2}} - \frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} \\
&= \frac{(x^2+y^2)^{\frac{1}{2}}(x^2+y^2)^{\frac{1}{2}} - x^2}{(x^2+y^2)^{\frac{1}{2}}} \\
&= \frac{x^2+y^2-x^2}{(x^2+y^2)^{\frac{1}{2}}} \\
&= \frac{y^2}{(x^2+y^2)^{\frac{1}{2}}}
\end{aligned}$$

50. $\frac{x^{\frac{1}{2}}+x}{x^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}} + \frac{x}{x^{\frac{1}{2}}} = 1 + x^{\frac{1}{2}}$

51.

$$\begin{aligned}
(u^{\frac{1}{3}} - v^{\frac{1}{3}})(u^{\frac{2}{3}} + (uv)^{\frac{1}{3}} + v^{\frac{2}{3}}) &= u^{\frac{1}{3}}u^{\frac{2}{3}} + u^{\frac{1}{3}}(uv)^{\frac{1}{3}} + u^{\frac{1}{3}}v^{\frac{2}{3}} - v^{\frac{1}{3}}u^{\frac{2}{3}} - v^{\frac{1}{3}}(uv)^{\frac{1}{3}} - v^{\frac{1}{3}}v^{\frac{2}{3}} \\
&= u - v
\end{aligned}$$

52. $0.00317 = 3.17 \times 10^{-3}$

53. $2.15 \times 10^7 \times 3.54 \times 10^{-1} = 2.15 \times 3.54 \times 10^{7-1} = 7.611 \times 10^6$

54. $3.47 \times 10^{17} \times 7.4 \times 10^{-3} = 3.47 \times 7.4 \times 10^{17-3} = 25.678 \times 10^{14} = 2.5687 \times 10^{15}$

55. $(2.7 \times 10^{65})^{\frac{1}{3}} = (0.27 \times 10^{66})^{\frac{1}{3}} = (0.27)^{\frac{1}{3}} \times (10^{66})^{\frac{1}{3}} \approx 0.646 \times 10^{66\times\frac{1}{3}} = 6.46 \times 10^{22}$

56. $5.98 \times 10^6 - 3.7 \times 10^5 = 5980000 - 370000 = 5610000 = 5.61 \times 10^6$

57. $\frac{(3.8\times10^{27})}{(2.45\times10^{-8})} = \frac{3.8}{2.45} \times 10^{27-(-8)} = 1.55 \times 10^{35}$

58. $(-1.7)^{\frac{1}{8}}$ is undefined.

59. $(-3)^{\frac{2}{19}}$ is defined.

60. $(-4.8)^{-\frac{6}{14}} = (-4.8)^{-\frac{3}{7}}$ is defined.

61. $(\pi)^{\sqrt{2}}$ is defined

62. $(-\pi)^{-\frac{8}{14}} = (-\pi)^{-\frac{4}{7}}$ is defined.

4.2 Solutions to Exercises from Section 2

1. The graphs of $y = 2.5^x$ and $y = 5^{-x}$ appear below. In both cases the graph intercepts the y-axis at $y = 1$. In neither case does the graph intercept the x-axis, though the graph does get extremely close to the x-axis in both cases.

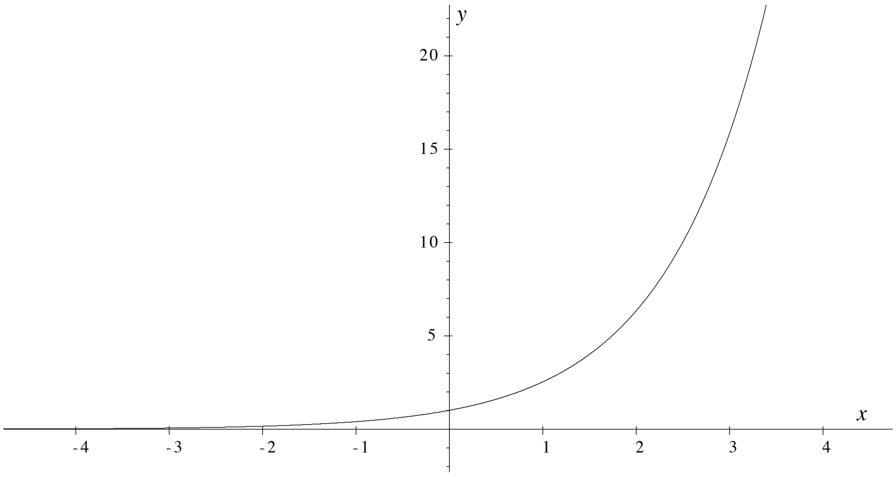

Figure 8: Graph of $y = 2.5^x$.

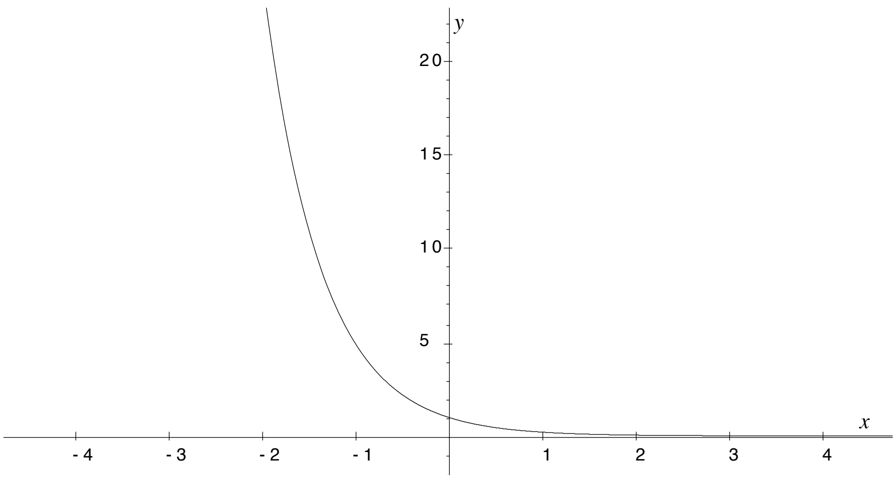

Figure 9: Graph of $y = 5^{-x}$.

2. Remember that the function $f(x) = b^x$ is increasing if $b > 1$ and is decreasing if $b < 1$.

a. $f(x) = 2.7^x$ is increasing since $2.7 > 1$.

b. $f(x) = (\frac{1}{2.7})^{-x} = 2.7^x$, so this function is also increasing.

c. $f(x) = 3^{-x} = (\frac{1}{3})^x$ is decreasing since $\frac{1}{3} < 1$.

d. $f(x) = 0.22^x$ is decreasing since $0.22 < 1$.

3. Again, remember that the function $f(x) = b^x$ is increasing if $b > 1$ and is decreasing if $b < 1$.

a. $f(x) = (\frac{5}{3})^x$ is increasing because $\frac{5}{3} > 1$.

b. $f(x) = (\frac{5}{3})^{-x} = (\frac{3}{5})^x$ is decreasing because $\frac{3}{5} < 1$.

c. $f(x) = (\frac{3}{5})^{-x} = (\frac{5}{3})^x$ is increasing because $\frac{5}{3} > 1$.

d. $f(x) = (\frac{3}{5})^x$ is decreasing because $\frac{3}{5} < 1$.

4. The graphs are drawn in Figures 21 and 22 below. Notice that the graph of $f(x) = 2.9^x$ is very close to the graph of $f(x) = 3^x$, and similarly for the other pair of graphs.

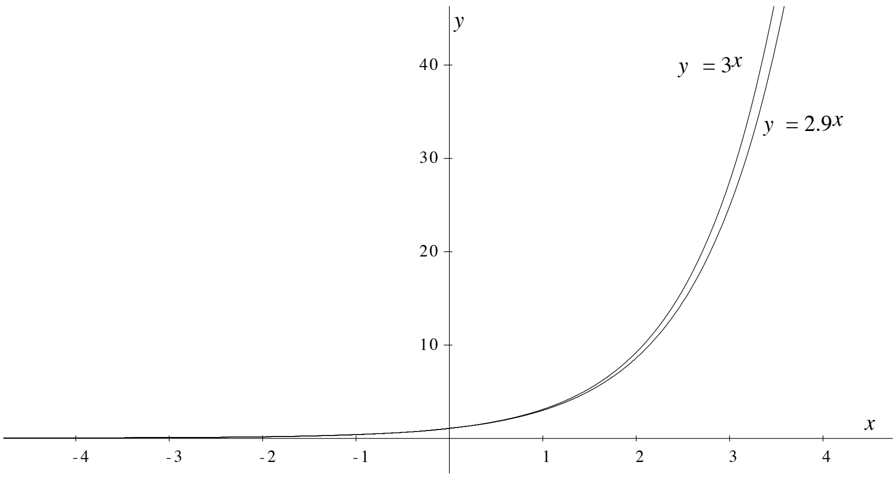

Figure 10: Graphs of $y = 3^x$ and $y = 2.9^x$.

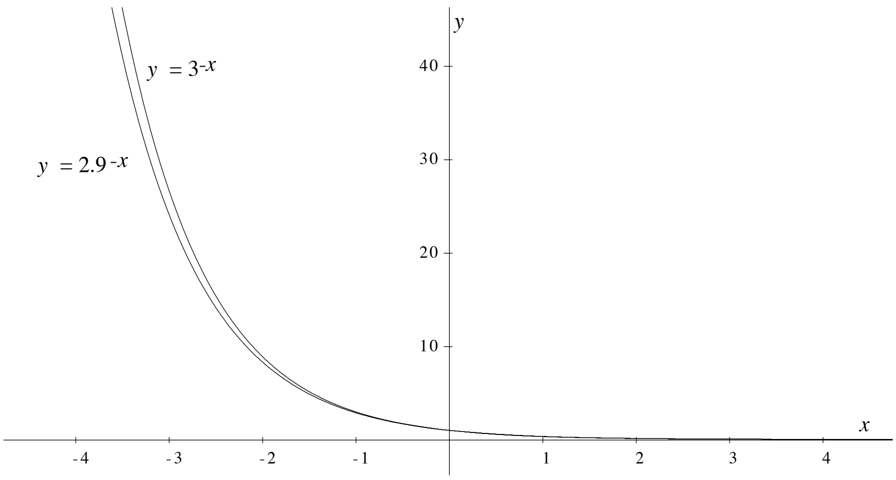

Figure 11: Graphs of $y = 3^{-x}$ and $y = 2.9^{-x}$.

5. On my calculator I get $e^{1.09861} = 2.999993$. Now

$$\begin{aligned} e^{1.09861x} &= (e^{1.09861})^x \\ &\approx 3^x \end{aligned}$$

Thus the functions 3^x and $e^{1.09861x}$ agree very closely with each other.

4.3 Solutions to Exercises from Section 3.

1. **a.** $\log_{10} 10000 = \log_{10} 10^4 = 4$

b. $\log_{10} \frac{1}{100} = \log_{10} 10^{-2} = -2$

c. $\log_{10} 0.001 = \log_{10} 10^{-3} = -3$

d. $\log_{10} 10^{12.3} = 12.3$

e. $\log_{10} \sqrt{10} = \log_{10} 10^{\frac{1}{2}} = \frac{1}{2}$

f. $\log_{10} 10\sqrt[4]{10} = \log_{10} 10 \times 10^{\frac{1}{4}} = \log_{10} 10^{1+\frac{1}{4}} = \frac{5}{4}$

g. $\log_{10}(\frac{1}{1000})\sqrt[3]{10} = \log_{10} 10^{-3+\frac{1}{3}} = -\frac{8}{3}$

h. $\log_{10} \frac{1}{0.001} = \log_{10} 10^3 = 3$

2. **a.** $10^{\log_{10} 37.23} = 37.23$ by rule B.

b. $\log_{10} 10^{x^2y} = x^2y$ by Rule A.

c. $10^{\log_{10}(10^x)} = 10^x$, since $\log_{10}(10^x) = x$ by rule A.

d. $\log_{10} \sqrt{10}^{\frac{x}{2y}} = \log_{10}(10^{\frac{1}{2}})^{\frac{x}{2y}} = \log_{10} 10^{\frac{1}{2}\times\frac{x}{2y}} = \frac{x}{4y}$ by rule A.

e. $10^{10^{\log_{10} x}} = 10^x$, since $10^{\log_{10} x} = x$ by rule B.

f. $\log_{10} 10^{\frac{x+y}{z}} = \frac{x+y}{z}$ by rule A.

g. $10^{\log_{10} \frac{3xy}{z}} = \frac{3xy}{z}$ by rule B.

h. $\log_{10} 10^{10^{2x}} = 10^{2x}$ by rule A.

3. **a.** $\log_{10} x^3 - 2.5\log_{10} y = \log_{10} x^3 - \log_{10} y^{2.5} = \log_{10} \frac{x^3}{y^{2.5}}$

b. $\log_{10} 6 + \log_{10} x^{-2} = \log_{10} 6x^{-2}$

c. $5\log_{10} 3x - 4\log_{10}(xy+z^2) = \log_{10}(3x)^5 - \log_{10}(xy+z^2)^4 = \log_{10} \frac{(3x)^5}{(xy+z^2)^4}$

d. $2\log_{10} xy + 3\log_{10}(z^2-y^2) = \log_{10}(xy)^2 + \log_{10}(z^2-y^2)^3 = \log_{10}(xy)^2(z^2-y^2)^3$

e. $\log_{10}(x+y) - 3\log_{10} 4 = \log_{10}(x+y) - \log_{10} 4^3 = \log_{10} \frac{x+y}{64}$

f. $\log_{10} xy - 1.7\log_{10} y^2 = \log_{10} xy - \log_{10}(y^2)^{1.7} = \log_{10} \frac{xy}{y^{3.4}} = \log_{10} xy^{-2.4}$

4. **a.** $\log_2 2^x 2^{x+y} = \log_2 2^{x+x+y} = 2x+y$

b. $5^{\log_5 \frac{x+y}{3}} = \frac{x+y}{3}$

c. $\log_7 49^{uv} = \log_7(7^2)^{uv} = \log_7 7^{2uv} = 2uv$

d. $3^{\log_9 \frac{u}{3w}} = (9^{\frac{1}{2}})^{\log_9 \frac{u}{3w}} = 9^{\frac{1}{2}\times\log_9 \frac{u}{3w}} = 9^{\log_9(\frac{u}{3w})^{\frac{1}{2}}} = (\frac{u}{3w})^{\frac{1}{2}}$

5. **a.**

$$\begin{aligned} 2\log_3(x+y) - 3\log_3(xy) + \log_3 x^2 &= \log_3(x+y)^2 - \log_3(xy)^3 + \log_3 x^2 \\ &= \log_3 \frac{((x+y)^2x^2)}{(xy)^3} \end{aligned}$$

b.

$$\begin{aligned} \log_6 xy - 4\log_6(x+y) &= \log_6 xy - \log_6(x+y)^4 \\ &= \log_6 \frac{xy}{(x+y)^4} \end{aligned}$$

c.

$$\begin{aligned} 4\log_{17} xy^2 + \log_{17}(x^2+y^2) - 2.5\log_{17} x &= \log_{17}(xy^2)^4 + \log_{17}(x^2+y^2) - \log_{17} x^{2.5} \\ &= \log_{17} \frac{x^4y^8(x^2+y^2)}{x^{2.5}} \end{aligned}$$

6. **a.** $\log_3 17 = \log_3 10 \times \log_{10} 17 = \frac{\log_{10} 17}{\log_{10} 3} \approx \frac{1.2304}{0.4771} \approx 2.5789$

b. $\log_5 2 = \log_5 10 \times \log_{10} 2 = \frac{\log_{10} 2}{\log_{10} 5} \approx \frac{0.3010}{0.6990} \approx 0.4306$

c. $\log_{22} 14 = \log_{22} 10 \times \log_{10} 14 = \frac{\log_{10} 14}{\log_{10} 22} \approx \frac{1.1461}{1.3424} \approx 0.8538$

d. $\log_4 8 = \log_4 10 \times \log_{10} 8 = \frac{\log_{10} 8}{\log_{10} 4} \approx \frac{0.9031}{0.6021} \approx 1.5$ (in fact $\log_4 8$ is exactly 1.5 because $8 = 4^{\frac{3}{2}}$)

7. **a.** $y = 10^x = (e^{\ln 10})^x = e^{(\ln 10)x}$

b. $7.5^{-x} = (e^{\log_e 7.5})^{-x} = e^{-(\log_e 7.5)x}$

c. $4^{-x} = (e^{\log_e 4})^{-x} = e^{-(\log_e 4)x}$

d. $(\frac{1}{4})^x = (e^{\log_e \frac{1}{4}})^x = e^{-(\log_e 4)x}$

8. $\log_{10} 10^{-19} = -19$

9. $\log_e e\sqrt[5]{e} = \log_e e^{\frac{6}{5}} = \frac{6}{5}$

10. $\log_2 16 = \log_2 2^4 = 4$

11. $\log_{17} \frac{17^3}{\sqrt{17}} = \log_{17} 17^{3-\frac{1}{2}} = \frac{5}{2}$

12. $\ln \frac{e^2}{e^{21}} = \ln e^{2-21} = -19$

13. $\frac{\ln e^7}{\log_{11} 121} = \frac{7}{\log_{11} 11^2} = \frac{7}{2}$

14. $5^{\log_5 32.7} = 32.7$

15. $e^{\ln \frac{9}{2}} = \frac{9}{2}$

16. $e^{\ln \sqrt[3]{27}} = \sqrt[3]{27} = 3$

17. $\log_{10} \frac{100x^2}{9y} = \log_{10} 100 + \log_{10} x^2 - \log_{10} 9y = 2 + 2\log_{10} x - \log_{10} 9 - \log_{10} y$

18. $\ln(\frac{xy^{-3}}{e^{1.37}}) = \ln x - 3\ln y - \ln e^{1.37} = \ln x - 3\ln y - 1.37$

19. $\log_4(\frac{4^{-1.3}z^7}{x^2y^3}) = -1.3 + 7\log_4 z - 2\log_4 x - 3\log_4 y$

20. $\log_3 \frac{x^3y^2}{27z^{\frac{1}{2}}} = 3\log_3 x + 2\log_3 y - \log_3 27 - \frac{1}{2}\log_3 z = 3\log_3 x + 2\log_3 y - 3 - \frac{1}{2}\log_3 z$

21. $\ln(e^{-2.4}x^6) = -2.4 + 6\ln x$

22. $\log_5 \frac{125x^3}{0.2y^2} = \log_5 5^3 + 3\log_5 x - \log_5 \frac{1}{5} - 2\log_5 y = 4 + 3\log_5 x - 2\log_5 y$

23. $3\log_2 x + \log_2 30 + \log_2 y - \log_2 w = \log_2 \frac{30x^3y}{w}$

24. $2\ln x - \ln y + a\ln w = \ln x^2 - \ln y + \ln w^a = \ln \frac{x^2 w^a}{y}$

25. $12(\ln x + \ln y) = \ln(xy)^{12}$

26. $\log_3 e \times \ln 81 + \log_3 5 \times \log_5 w = \log_3 81 + \log_3 w = 4 + \log_3 w$

27. $\log_7 10 \times \log_{10} x^2 - \log_7 49x = \log_7 x^2 - \log_7 49 - \log_7 x = -2 + \log_7 \frac{x^2}{x} = -2 + \log_7 x$

28. $\log_{10} 0.1 \times \log_6 x - 2\log_6 y + \log_6 4 \times \log_4 e = -1 \times \log_6 x - \log_6 y^2 + \log_6 e = \log_6 \frac{e}{xy^2}$

29. $\log_5 e = \frac{1}{\log_e 5} \approx \frac{1}{1.6094} \approx 0.6213$

30. $\log_5 7 = \log_5 e \times \log_e 7 = \frac{\log_e 7}{\log_e 5} \approx \frac{1.9459}{1.6094} \approx 1.2091$

31. $\log_5 7^2 = \log_5 e \times 2\log_e 7 = \frac{2\log_e 7}{\log_e 5} \approx 2 \times \frac{1.9459}{1.6094} \approx 2.4182$

32. $\log_{49} 5 = \log_{49} e \times \log_e 5 = \frac{\log_e 5}{\log_e 49} = \frac{\log_e 5}{\log_e 7^2} \approx \frac{1.6094}{2\times 1.9459} \approx 0.4135$

33. $\log_{49} 25 = \log_{49} e \times \log_e 5^2 = \frac{\log_e 5^2}{\log_e 7^2} \approx \frac{2\times 1.6094}{2\times 1.9459} \approx 0.8271$

34. $\log_e 25 = \log_e 5^2 \approx 2 \times \log_e 5 \approx 2 \times 1.6094 \approx 3.2188$

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