Matrix multiplication

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Multiplying matrices

We can multiply matrices A and B together to form the product AB provided the number of columns in A **equals** the number of rows in B.

If $A = \begin{bmatrix} 4 & -1 & 3 \\ 1 & -2 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -5 \\ -1 & -4 \\ 0 & -1 \end{bmatrix}$

then we can define AB as A has three columns and B has three rows.

Multiplying matrices

If $A = \begin{bmatrix} 4 & -1 & 3 \\ 1 & -2 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -5 \\ -1 & -4 \end{bmatrix}$

then AB is not defined as A is a 2×3 matrix and B is a 2×2 matrix; the number of columns of A does not equal the number of rows of B.

On the other hand, the product BA **is** defined as the number of columns of B, 2, does equal the number of rows of A.

This tells us something very important; order matters!!

In most cases $AB \neq BA$. Here AB is not defined whereas BA is.

How to multiply matrices

In general, if A is a $m \times n$ matrix and B is a $n \times p$ matrix, the product AB will be a $m \times p$ matrix.

Let $C = AB$. It is a $m \times p$ matrix.

Recall that the entry in the *ith* row and *jth* column of C, ie the $(i, j)th$ entry of C, is called c_{ij}.

The entry c_{ij} is the product of the *ith* row of A and the *jth* column of B as follows:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + a_{i4}b_{4j} + \cdots + a_{in}b_{nj}.$$

Example of multiplying matrices 1

That probably looked a bit complicated so we will go through an example.

Let $A = \begin{bmatrix} 0 & -5 \\ -1 & -4 \\ 6 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix}$

A is a 3×2 matrix and B is a 2×2, so AB is defined.

If $C = AB$ is then $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$ is a 3×2 matrix.

Example of multiplying matrices 2

$$\begin{bmatrix} 0 & -5 \\ -1 & -4 \\ 6 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

For the first entry c_{11} we multiple the **first** row of A with the **first** column of B as follows:

$$\begin{bmatrix} 0 & -5 \\ . & . \\ . & . \end{bmatrix} \times \begin{bmatrix} 0 & . \\ -3 & . \end{bmatrix} = \begin{bmatrix} 0 \times 0 + -5 \times -3 & . \\ . & . \\ . & . \end{bmatrix}$$

ie $c_{11} = a_{11} \times b_{11} + a_{12} \times b_{21} = 15$.

Example of multiplying matrices 3

$$\begin{bmatrix} 0 & -5 \\ -1 & -4 \\ 6 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 15 & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

For the entry c_{12} we multiple the **first** row of A with the **second** column of B as follows:

$$\begin{bmatrix} 0 & -5 \\ . & . \\ . & . \end{bmatrix} \times \begin{bmatrix} . & -1 \\ . & 2 \end{bmatrix} = \begin{bmatrix} . & 0 \times -1 + -5 \times 2 \\ . & . \\ . & . \end{bmatrix}$$

ie $c_{12} = a_{11} \times b_{12} + a_{12} \times b_{22} = -10$.

Example of multiplying matrices 4

$$\begin{bmatrix} 0 & -5 \\ -1 & -4 \\ 6 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 15 & -10 \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

For the entry c_{21} we multiple the **second** row of A with the **first** column of B as follows:

$$\begin{bmatrix} . & . \\ -1 & -4 \\ . & . \end{bmatrix} \times \begin{bmatrix} 0 & . \\ -3 & . \end{bmatrix} = \begin{bmatrix} . & . \\ -1 \times 0 + -4 \times -3 & . \\ . & . \end{bmatrix}$$

ie $c_{21} = a_{21} \times b_{11} + a_{22} \times b_{21} = 12$.

Example of multiplying matrices 5

So, to multiply two matrices we systematically work out each entry in this way, starting with the first entry.

$$\begin{bmatrix} 0 & -5 \\ -1 & -4 \\ 6 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} 15 & -10 \\ 12 & -7 \\ 6 & -10 \end{bmatrix}$$

Click to see how we get the other entries.

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Properties of matrix multiplication

Let A, B and C be matrices of dimensions for which the following expressions make sense, and let λ be a scalar. Then,

$$
\begin{aligned}
A(BC) &= (AB)C \\
A(B+C) &= AB+AC \\
(A+B)C &= AC+BC \\
\lambda(AB) &= (\lambda A)B = A(\lambda B) \\
\text{Note also that} \quad 0A &= 0 = A0
\end{aligned}
$$

The 0 in the last property could mean the scalar zero or the (appropriate) zero matrix. For example,

$$
0\begin{bmatrix} 2 & -5 \\ -1 & -4 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & -4 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$