Properties of determinants

Jackie Nicholas
Mathematics Learning Centre
University of Sydney

Property 1

If a square matrix A has a row (or column) of zeros, then $|A| = 0$.

Let $A = \begin{bmatrix} 3 & 0 & 2 \\ -1 & 0 & 5 \\ -1 & 0 & -2 \end{bmatrix}$, then

$$\begin{aligned} \begin{vmatrix} 3 & 0 & 2 \\ -1 & 0 & 5 \\ -1 & 0 & -2 \end{vmatrix} &= 0 \begin{vmatrix} -1 & 5 \\ -1 & -2 \end{vmatrix} + 0 \begin{vmatrix} 3 & 2 \\ -1 & -2 \end{vmatrix} + 0 \begin{vmatrix} 3 & 2 \\ -1 & 5 \end{vmatrix} \\ &= 0. \end{aligned}$$

Property 2

If a square matrix A has two identical rows (or columns), then $|A| = 0$.

Let $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix}$, then

$$\begin{aligned} \begin{vmatrix} 3 & 1 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{vmatrix} &= 3\begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} -1 & -2 \\ -1 & -2 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix} \\ &= 3(-2+2) - 1(2-2) + 1(-1+1) \\ &= 0. \end{aligned}$$

Property 3

If all entries below (or above) the main diagonal of a square matrix A are zero, then $|A|$ is equal to the product of the main diagonal entries .

Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix}$, then expanding down the first column

$$\begin{aligned} \begin{vmatrix} 3 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{vmatrix} &= 3\begin{vmatrix} 1 & -2 \\ 0 & -2 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} \\ &= 3(1)(-2). \end{aligned}$$

Note that for any identity matrix, $|I_n| = 1$.

Property 4

If a matrix B is obtained from a square A by adding a multiple of one row (or column) to another row (or column), then $|B| = |A|$.

Let $A = \begin{bmatrix} -1 & 1 & 1 \\ 6 & 1 & -2 \\ 4 & 0 & -2 \end{bmatrix}$, and $B = \begin{bmatrix} 7 & 1 & -3 \\ 6 & 1 & -2 \\ 4 & 0 & -2 \end{bmatrix}$ then

B is the matrix obtained from A by adding $2\times$ row 3 to row 1.

$$\begin{bmatrix} 7 & 1 & -3 \\ 6 & 1 & -2 \\ 4 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -1+8 & 1+0 & 1-4 \\ 6 & 1 & -2 \\ 4 & 0 & -2 \end{bmatrix} \text{ so } |B| = |A|.$$

Property 5

If a matrix B is obtained from a A by interchanging two rows (or columns), then $|B| = -|A|$.

Let $A = \begin{bmatrix} -1 & 5 & 2 \\ 1 & 0 & -4 \\ 4 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 0 & -4 \\ -1 & 5 & 2 \end{bmatrix}$.

$$\begin{aligned} |A| &= -1\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} - 5\begin{vmatrix} 1 & -4 \\ 4 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} \\ &= -1(0+4) - 5(-1+16) + 2(1-0) = -77. \end{aligned}$$

$$\begin{aligned} |B| &= 4\begin{vmatrix} 0 & -4 \\ 5 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -4 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ -1 & 5 \end{vmatrix} \\ &= 4(0+20) - 1(2-4) - 1(5) = 77. \end{aligned}$$

Property 6

If a matrix B is obtained from a A by multiplying all the entries in one row (or column) by a scalar, k, then $|B| = k|A|$.

$$\text{Let } A = \begin{bmatrix} -1 & 5 & 2 \\ 1 & 0 & -4 \\ 4 & 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 10 & 2 \\ 1 & 0 & -4 \\ 4 & 2 & -1 \end{bmatrix},$$

so the entries in the second column of B are twice that of A.

$$\begin{aligned} \begin{vmatrix} -1 & 10 & 2 \\ 1 & 0 & -4 \\ 4 & 2 & -1 \end{vmatrix} &= -1\begin{vmatrix} 0 & -4 \\ 2 & -1 \end{vmatrix} - 10\begin{vmatrix} 1 & -4 \\ 4 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 4 & 2 \end{vmatrix} \\ &= (-1)(8) - 10(15) + 2(2) = -154 = 2(-77). \end{aligned}$$

Property 7

If A and B are $n \times n$ matrices, then $|AB| = |A| \times |B|$.

$$\text{Let } A = \begin{bmatrix} -1 & 5 \\ 4 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix},$$

then

$$|AB| = |A| \times |B| = \begin{vmatrix} -1 & 5 \\ 4 & 2 \end{vmatrix} \times \begin{vmatrix} -1 & 2 \\ 1 & -4 \end{vmatrix} = (-22)(2) = -44.$$

A consequence of this property and $|I_n| = 1$ is:

If A^{-1} exists then $|A^{-1}| \times |A| = |A^{-1}A| = |I_n| = 1$,
so

$$|A^{-1}| = \frac{1}{|A|}.$$