Mathematics Learning Centre

The University of Sydney

Revising algebra skills

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1 Revision of Algebraic Skills

1.1 Why use algebra?

Algebra is used extensively throughout the sciences and economics. It allows us to examine relationships between two or more variables.

For example,

$$S_n = P(1 + r)^n$$

gives the formula for the amount, S_n, available on maturity when a principal, P, is invested for n years at a rate of r per annum.

Let's suppose we have $10,000$ to invest for 3 years at an interest rate of 5.25% per annum. Then,

$$S_3 = 10000(1 + 0.0525)^3 = 11659.13.$$

So, at the end of 3 years we will have \$11,659.13 in the account.

One advantage of expressing this formula algebraically is that we can vary our values of P, r and n and use the formula again and again. Another advantage of having the algebraic formula, is that we can reconfigure it and ask a different question.

For example, suppose we are have a sum of money left to us as an inheritance. We want to spend some now but also set aside enough to buy a new car in 3 years time. How much money would we need to put into a savings account earning 6% per annum now in order to have \$25,000 in 3 years time.

In this example, we know $S_n = 25000$, $r = 0.06$ and $n = 3$, but we do not know P. We can find P by reconfiguring the equation,

$$P = \frac{S_n}{(1 + r)^n}.$$

That is

$$P = \frac{25000}{(1 + 0.06)^3} = \frac{25000}{1.06^3} = \frac{25000}{1.191016} = 20990.48.$$

So, we would need to put \$20990.48 into an account now in order to have \$25,000 available for a car in 3 years time.

We used algebra to work this out.

In this lecture we will revise some general algebra skills.

1.2 General rules of algebra

1.2.1 Some general rules

In this section we present some general rules of algebra, with an example where appropriate. We will not attempt to justify them here. If you are interested in this or want more examples to practise then there are many elementary algebra books available.

- $x + x + x = 3 \times x = 3x$.

- $x \times x = x^2$. So, $3ab^3 = 3 \times a \times b^3 = 3 \times a \times b \times b \times b$.

The following rules hold where n and m are real numbers and for all x where x^n and x^m are defined.

- $x^n \times x^m = x^{n+m}$. For example, $x^3 \times x^2 = x^5$.
- $x^n \div x^m = x^{n-m}$. For example, $p^6 \div p^4 = p^{6-4} = p^2$.
- $(x^n)^m = x^{nm}$. For example, $(x^4)^2 = x^{4\times 2} = x^8$.
- $x^0 = 1$, for any x other than 0, since $x^n \div x^n = x^{n-n} = x^0 = 1$.
- $x^{-n} = \dfrac{1}{x^n}$. This is a consequence of the above rule for division, as $x^3 \div x^5 = x^{3-5} = x^{-2}$, but $x^3 \div x^5 = \dfrac{x^3}{x^5} = \dfrac{1}{x^2}$.
- $\sqrt[n]{x} = \dfrac{1}{x^n}$. For example, $\sqrt{x} = x^{\frac{1}{2}}$.

1.2.2 Collecting like terms

Express $4p - 3q + r - 2q + p - 5r$ in its simplest form.

Here we can collect like terms together and simplify as follows:

$$4p - 3q + r - 2q + p - 5r = 4p + p - 3q - 2q + r - 5r = 5p - 5q - 4r.$$

Note that the signs can be thought of as belonging to the pronumeral, p, q or r that follows the sign.

Example Express $5xy^2 - 8x^2y + 10xy - 3x^2y - 3xy$ in its simplest form.

Solution We can rewrite the equation as follows:

$$5xy^2 - 8x^2y + 10xy - 3x^2y - 3xy = 5xy^2 - 8x^2y - 3x^2y + 10xy - 3xy = 5xy^2 - 11x^2y + 7xy.$$

1.2.3 Removing brackets

There are many instances where we use brackets in mathematics for grouping parts of expressions. For example in the formula

$$S_n = P(1 + r)^n,$$

the brackets tell us to add 1 and r first, $(1 + r)$, and then raise the result to the power of n.

There are rules for their removal and some traps to avoid.

Simple brackets

In an expression like $(2x - 3) + (x + 1)$ the brackets can be removed straightforwardly to get $2x - 3 + x + 1 = 3x - 2$.

However in an expression like $(2x-3)-(x+1)$, we must realise that we are substracting the whole expression $x+1$ and take this into account when we remove the brackets.

$$(2x-3)-(x+1) = 2x-3-x-1 = x-4.$$

Notice that when we removed the bracket around the second term, we got $-x$ and -1.

The distributive law

An expression like $5(x+1)$ means that we take x and add 1 and multiply the result by 5.

When removing the bracket, we multiply each of the symbols inside the bracket by 5.

That is $5(x+1) = 5\times x + 5\times 1 = 5x+5$.

This is an application of the distributive law, $a(x+y) = ax+ay$.

Examples

Remove the brackets and simplify the following expressions:

1. $6(3a-1)+4(a-5)$,

2. $-2(4b-1)-(b+3)$,

3. $3a(a-b)+b(a+3)$.

Solutions

1. $6(3a-1)+4(a-5) = 18a-6+4a-20 = 22a-26$.

2. $-2(4b-1)-(b+3) = -8b+2-b-3 = -9b-1$.

3. $3a(a-b)+b(a+3) = 3a^2-3ab+ab+3b = 3a^2-2ab+3b$.

We can use the distributive law to multiply out the product of two (or more) brackets. For example, expand $(y+3)(5y-2)$. To use the distributive law here we distribute $(y+3)$ through the second bracket to get

$$(y+3)(5y-2) = (y+3)5y + (y+3)(-2) = 5y^2+15y-2y-6 = 5y^2+13y-6.$$

This is the same as taking each term in the first bracket and multiplying through the second bracket term by term as follows:

$$(y+3)(5y-2) = 5y^2-2y+15y-6 = 5y^2+13y-6.$$

We can then extend this to more complex expressions.

For example,

$$(2p-4)(p^2+3p-1) = 2p^3+6p^2-2p-4p^2-12p+4 = 2p^3+2p^2-14p+4.$$

1.2.4 Factorising

Extracting factors from algebraic expressions is the reverse process of expanding brackets. The distributive law says that $a(x+y) = ax + ay$.

We can turn it around and write it as $ax + ay = a(x+y)$.
In this case we say that $ax + ay$ is expressed as a product of factors a and $(x+y)$.

For example, $3p^2 - 12pq = 3p(p-4q)$ and $66 - 36ab = 6(11-6ab)$.
In each case we looked for a common factor in each term of the expression and extracted it as a factor. The other factor is what is left when we divide each term by that common factor.
It is always a good idea to check your answer by multiplying out the bracket again in your head.

Difference of two squares
The following equation can be using to factorise the difference of two squares,

$$x^2 - y^2 = (x+y)(x-y).$$

For example, $a^2 - 9b^2 = (a+3b)(a-3b)$.

1.2.5 Algebraic fractions

The rules of fractions also apply to algebraic fractions.
For addition and subtraction, write each fraction as an equivalent fraction using a common denominator if necessary.

$$\frac{a}{4} + \frac{3a}{4} = \frac{a+3a}{4} = \frac{4a}{4} = a.$$

$$\frac{x}{3} - \frac{x}{4} = \frac{4x - 3x}{12} = \frac{x}{12}.$$

For multiplication, multiply the numerators together and multiply the denominators together and then cancel any common factors (or cancel the common factors first).

$$\frac{3m}{n} \times \frac{2n}{l} = \frac{3m \times 2n}{n \times l} = \frac{6mn}{nl} = \frac{6m}{l}.$$

For division, invert the fraction in the denominator and multiply. (You are in effect multiplying both the numerator and the denominator by the reciprocal of the donominator.)

$$\begin{aligned}\frac{2p}{3q} \div \frac{8p}{9q} &= \frac{2p}{3q} \times \frac{9q}{8p} \\ &= \frac{2p \times 9q}{3q \times 8p} \\ &= \frac{18pq}{24pq} \\ &= \frac{3}{4}.\end{aligned}$$

1.3 Substituting into formulae

We are often in the situation where we need to substitute values into a formula. In each case we replace the variable with its value and the form of the equation will determine the way we evaluate it. We will give some examples here from economics or statistics to illustrate this.

1. A demand equation is given by $p = 600 - 2q$ where p is the unit price of a good and q is number of units of the good that the consumers will demand at that price during a given time period. Find the value of p when $q = 100$.

Here we can substitute $q = 100$ straight into the equation as follows:

$$p = 600 - 2q = 600 - 2(100) = 600 - 200 = 400.$$

Notice that we multiplied $q = 100$ by 2 and then subtracted the result from 600. This is what the formula 'told' us to do in this situation.

2. A firm's total cost of production (TC) is given as a function of output q by the equation

$$TC = q^3 - 20q^2 + 220q.$$

Find the total cost of production when $q = 10$.

Again we can substitute into the equation as follows:

$$TC = q^3 - 20q^2 + 220q = (10)^3 - 20(10)^2 + 220(10) = 1000 - 2000 + 2200 = 1200.$$

3. When we take a sample of size n from an (infinitely) large population, the formula for a 95% confidence interval for population proportion is

$$p_s \pm 1.96\sqrt{\frac{p_s(1-p_s)}{n}}.$$

If $p_s = 0.4$, and $n = 100$, find the 95% confidence interval for population proportion.

$$p_s \pm 1.96\sqrt{\frac{p_s(1-p_s)}{n}} = 0.4 \pm 1.96\sqrt{\frac{(0.4)(0.6)}{100}} = 0.4 \pm 1.96\sqrt{0.0024} = 0.4 \pm 0.0960.$$

4. An exponential model with a seasonal component is fitted to time series data and is given by:

$$\widehat{Y} = b_0 b_1^{X_i} b_2^{Q_1} b_3^{Q_2} b_4^{Q_3}.$$

Find $\widehat{Y}$ when $b_0 = 6$, $b_1 = 0.5$, $b_2 = 0.1$, $b_3 = 0.2$, $b_4 = 0.2$, $X_i = 10$, $Q_1 = Q_3 = 0$ and $Q_2 = 1$.

Here we have

$$\widehat{Y} = b_0 b_1^{X_i} b_2^{Q_1} b_3^{Q_2} b_4^{Q_3} = 6(0.5^{10})(0.1^0)(0.2^1)(0.2^0) = 6(0.000976)(0.2) = 0.001172.$$

Note that $0.1^0 = 0.2^0 = 1$. (Don't worry about what each of the symbols represent.)

1.4 Equations

It is often necessary to solve equations in economics and statistics. For example, we may be given a demand equation $p = 600 - 2q$ but need to rewrite it as an equation $q = f(p)$, ie so q is the subject. In this case

$$\begin{aligned} p &= 600 - 2q \\ p - 600 &= -2q \\ (-1)(p - 600) &= 2q \\ 600 - p &= 2q \\ q &= \frac{600 - p}{2} \\ &= 300 - \frac{p}{2}. \end{aligned}$$

How do we know what to do at each stage? One way to approach this is to write down the operations we went through to get from q to p, and then backtrack to get from p to q. (If you have another method that you learned in school that gives you the correct answer, then use that method.)

$$q \underbrace{\longrightarrow}_{\times 2} 2q \underbrace{\longrightarrow}_{\times -1} -2q \underbrace{\longrightarrow}_{+600} 600 - 2q = p.$$

To get from p back to q we do the operation that undoes each of these operations in the reverse order, ie we do the last one first. Of course, to keep an equivalent equation we do each operation to both sides of the equation.

$$\begin{aligned} p &= 600 - 2q & & \\ p - 600 &= -2q & & \text{subtract } 600 \\ (-1)(p - 600) &= 2q & & \text{multiply by } -1 \\ 600 - p &= 2q & & \\ q &= \frac{600 - p}{2} & & \text{divide by } 2 \\ &= 300 - \frac{p}{2}. & & \end{aligned}$$

In statistics, the following formula is well known:

$$Z = \frac{X - \mu}{\sigma}.$$

We often wish to rewrite it so that it is in the form $X = \ldots$ in which case our analysis gives

$$X \underbrace{\longrightarrow}_{-\mu} X - \mu \underbrace{\longrightarrow}_{\div \sigma} \frac{X - \mu}{\sigma} = Z.$$

Working backwards undoing each operation we get:

$$\begin{aligned} Z &= \frac{X-\mu}{\sigma} & \\ Z\times\sigma &= X-\mu & \text{multiply by } \sigma \\ Z\sigma+\mu &= X. & \text{add } \mu \end{aligned}$$

So, $X = \mu + Z\sigma$.

Here is another formula from statistics:

$$SE(p) = \sqrt{\frac{p(1-p)}{n}}$$

If we want an expression where n is the subject, our analysis gives:

$$n \quad \underbrace{\longrightarrow}_{\text{take reciprocal}} \quad \frac{1}{n} \quad \underbrace{\longrightarrow}_{\times p(1-p)} \quad \frac{p(1-p)}{n} \quad \underbrace{\longrightarrow}_{\text{take square root}} \quad \sqrt{\frac{p(1-p)}{n}} = SE(p).$$

Note that the reciprocal of n is $\frac{1}{n}$, and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

Working backwards undoing each operation we get:

$$\begin{aligned} SE(p) &= \sqrt{\frac{p(1-p)}{n}} & \\ SE(p)^2 &= \frac{p(1-p)}{n} & \text{square both sides} \\ \frac{SE(p)^2}{p(1-p)} &= \frac{1}{n} & \text{divide by } p(1-p) \\ \frac{p(1-p)}{SE(p)^2} &= n. & \text{take reciprocal} \end{aligned}$$

So, $n = \dfrac{p(1-p)}{SE(p)^2}$.

1.5 Solving quadratic equations

Any equation of the form $p = aq^2 + bq + c$ where $a \neq 0$, b and c are constants, is a quadratic equation.

For example, $p = -4 + 5q - q^2$ is quadratic.

If we want to find a solution for the equation $aq^2 + bq + c = 0$ (ie $p = 0$) then we can use the (well known) formula:

$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(Note: If $b^2 - 4ac$ is a negative number, then there are no real solutions to the equation $aq^2 + bq + c = 0$.)

Example Solve the equation $-4 + 5q - q^2 = 0$.

Solution First of all, rewrite the equation so that the coefficient of q^2 is positive,

$$q^2 - 5q + 4 = 0.$$

Then,

$$\begin{aligned} q &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(4)}}{2} \\ &= \frac{5 \pm \sqrt{25 - 16}}{2} \\ &= \frac{5 \pm \sqrt{9}}{2} \\ &= \frac{5 \pm 3}{2} \end{aligned}$$

giving us two solutions, $q = \frac{5+3}{2} = 4$ and $q = \frac{5-3}{2} = 1$.

1.6 Exercises

1.6.1 General algebraic skills

1. Applying the general rules

Simplify the following expressions using the general rules:

a. $2p^3 \times 5p^2q$

b. $\dfrac{-4x^3}{8x^2}$

c. $(-2a^2)^4$.

2. Collecting like terms

Simplify the following expressions:

a. $3x^2 - 2xy - y^3 + 5xy + 4y^3$

b. $5 - 2ab^3 + 4ab^2 - a^3 - 8 - 5ab^2 + 6a^2b - 10$

c. $4pq - 6q^2 - 12 + 2p^3 - 8qp + q^2 - 6$.

3. Removing brackets

In the following exercises, remove the brackets and simplify if appropriate:

a. $(2p - 3q^2) - (4q^4 - 5p + 4)$

b. $2x(x^2 - y^2 - 2xy + x - y) + (x + y)(x - y)$

c. $(x - 2)(x + 3) - (x + 1)(1 - x)$.

4. Factorising

Factorise the following expression:

a. $3p^3q - 12pq + 36p^2q^2$

b. $ax + a - bx - b$

c. $9p^2 - 16q^2$.

5. Algebraic fractions

Simplify the following algebraic fractions

a. $\dfrac{pq}{p^2q^2 - 2pq}$

b. $1 - \dfrac{a - b}{a^2 - b^2}$

c. $\dfrac{x^2 - 2xy}{y^2 - y} \times \dfrac{y - 1}{x - 2y}$.

1.6.2 Substituting into formulae

1. The Variance Inflationary Factor (VIF_j) is given by the formula,

$$VIF_j = \frac{1}{(1 - R_j^2)}.$$

Evaluate VIF_j when $R_j = 0.8$.

2. A firm's profit, π, is given as a function of output, q, by the equation

$$\pi = 80q - 4q^2 - 200.$$

Find the value of π when $q = 10$.

3. The Geometric Mean (GM) is given by the formula,

$$GM = \sqrt[n]{X_1 \times X_2 \times \cdots \times X_n}.$$

Find the value of the Geometric Mean if $n = 6$ and $X_1 = 1.04$, $X_2 = 1.15$, $X_3 = 0.97$, $X_4 = 1.12$, $X_5 = 1.08$ and $X_6 = 1.14$.

1.6.3 Equations

1. $q = 400 - 2p$. Rewrite this formula making p the subject.

2.

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

is another formula from statistics. Rewrite this formula making $\overline{X}$ the subject.

3. The Variance Inflationary Factor (VIF_j) is given by the formula,

$$VIF_j = \frac{1}{(1 - R_j^2)}.$$

Assuming that $R_j \geq 0$, rewrite this equation making R_j the subject (ie as $R_j = \ldots$).

4. Solve the following quadratic equations using the quadratic formula.

a. $q^2 + q - 12 = 0$

b. $5x - 6x^2 - 1 = 0$

c. $80q - 4q^2 - 200 = 0$

1.7 Answers to exercises

1.7.1 General algebraic skills

1. Applying the general rules

a. $10p^5q$

b. $-\frac{x}{2}$

c. $16a^8$

2. Collecting like terms

a. $3x^2 + 3xy + 3y^2$

b. $-2ab^3 - ab^2 + 6a^2b - a^3 - 13$

c. $2p^3 - 4pq - 5q^2 - 18$

3. Removing brackets

a. $7p - 3q^2 - 4q^4 - 4$

b. $2x^3 - 2xy^2 - 4x^2y + 3x^2 - 2xy - y^2$

c. $2x^2 + x - 7$

4. Factorising

a. $3pq(p^2 + 12pq - 4)$

b. $(a - b)(x + 1)$

c. $(3p + 4q)(3p - 4q)$

5. Algebraic fractions

a. $\dfrac{1}{(pq-2)}$

b. $\dfrac{a+b-1}{a+b}$

c. $\dfrac{x}{y}$

1.7.2 Substituting into formulae

1. $VIF_j = 2.7778$ to 4 decimal places

2. $\pi = 200$

3. $GM = 1.0815$ to 4 decimal places

1.7.3 Equations

1. $p = 200 - \dfrac{q}{2}$

2. $\overline{X} = \mu + \dfrac{z\sigma}{\sqrt{n}}$

3. $R_j = \sqrt{1 - \dfrac{1}{VIF_j}}$

(Here we were told that R_j is positive, so we took the positive square root as our solution.)

4. **a.** $q = -4$ or $q = 3$

b. $x = \frac{1}{2}$ or $x = \frac{1}{3}$

c. $q = 17.0711$ or $q = 2.9289$