Mathematics Learning Centre

The University of Sydney

The rules of calculus

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1 How do we find derivatives (in practice)?

Differential calculus is a procedure for finding the exact derivative directly from the formula of the function, without having to use graphical methods. In practise we use a few rules that tell us how to find the derivative of almost any function that we are likely to encounter. In this section we will introduce these rules to you, show you what they mean and how to use them.

Warning! To follow the rest of these notes you will need feel comfortable manipulating expressions containing indices. If you find that you need to revise this topic you may find the Mathematics Learning Centre publication *Exponents and Logarithms* helpful.

1.1 Derivatives of constant functions and powers

Perhaps the simplest functions in mathematics are the constant functions and the functions of the form x^n.

Rule 1 If k is a constant then $\dfrac{d}{dx}k = 0$.

Rule 2 If n is any number then $\dfrac{d}{dx}x^n = nx^{n-1}$.

Rule 1 at least makes sense. The graph of a constant function is a horizontal line and a horizontal line has slope zero. The derivative measures the slope of the tangent, and so the derivative is zero.

How you approach Rule 2 is up to you. You certainly need to know it and be able to use it. However we have given no justification for why Rule 2 works! In fact in these notes we will give little justification for any of the rules of differentiation that are presented. We will show you how to apply these rules and what you can do with them, but we will not make any attempt to prove any of them.

Examples If $f(x) = x^7$ then $f'(x) = 7x^6$.

If $y = x^{-0.5}$ then $\frac{dy}{dx} = -0.5x^{-1.5}$.

$\dfrac{d}{dx}x^{-3} = -3x^{-4}$.

If $g(x) = 3.2$ then $g'(x) = 0$.

If $f(t) = t^{\frac{1}{2}}$ then $f'(t) = \frac{1}{2}t^{-\frac{1}{2}}$.

If $h(u) = -13.29$ then $h'(u) = 0$.

In the examples above we have used Rules 1 and 2 to calculate the derivatives of many simple functions. However we must not lose sight of what it is that we are calculating here. The derivative gives the slope of the tangent to the graph of the function.

For example, if $f(x) = x^2$ then $f'(x) = 2x$. To find the slope of the tangent to the graph of x^2 at $x = 1$ we substitute $x = 1$ into the derivative. The slope is $f'(1) = 2 \times 1 = 2$. Similarly the slope of the tangent to the graph of x^2 at $x = -0.5$ is found by substituting $x = -0.5$ into the derivative. The slope is $f'(-0.5) = 2 \times -0.5 = -1$. This is illustrated in Figure 1.

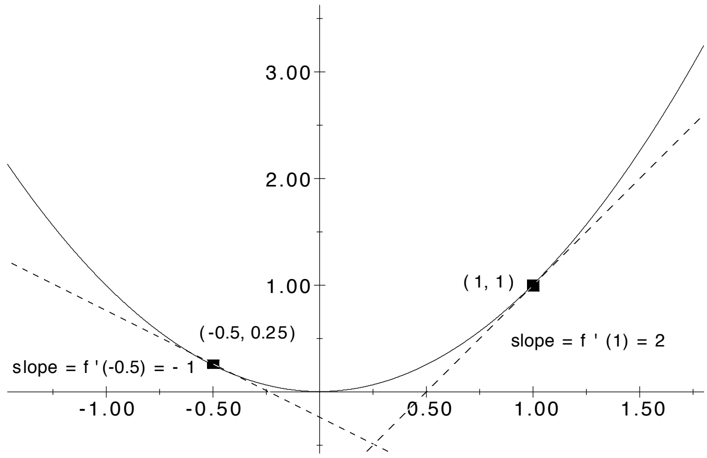

Figure 1: Slopes of tangents to the graph of $y = x^2$.

Example

Find the slope of the tangent to the graph of the function $g(t) = t^4$ at the point on the graph where $t = -2$.

Solution

The derivative is $g'(t) = 4t^3$, and so the slope of the tangent line at $t = -2$ is $g'(-2) = 4 \times (-2)^3 = -32$.

Example

Find the equation of the line tangent to the graph of $y = f(x) = x^{\frac{1}{2}}$ at the point $x = 4$.

Solution

$f(4) = 4^{\frac{1}{2}} = \sqrt{4} = 2$, so the coordinates of the point on the graph are $(4, 2)$. The derivative is

$$f'(x) = \frac{x^{-\frac{1}{2}}}{2} = \frac{1}{2\sqrt{x}}$$

and so the slope of the tangent line at $x = 4$ is $f'(4) = \frac{1}{4}$. We therefore know the slope of the line and we know one point through which the line passes.

Any non vertical line has equation of the form $y = mx + b$ where m is the slope and b the vertical intercept.

In this case the slope is $\frac{1}{4}$, so $m = \frac{1}{4}$, and the equation is $y = \frac{x}{4} + b$. Because the line passes through the point $(4, 2)$ we know that $y = 2$ when $x = 4$.

Substituting we get $2 = \frac{4}{4} + b$, so that $b = 1$. The equation is therefore $y = \frac{x}{4} + 1$.

Notice that in the examples above the independent variable is not always called x. We have also used u and t, and in fact we can and will use many different letters for the independent variable. Notice also that we might not stick to the symbol f to stand for function. Many other symbols are used. Some of the common ones are g and h. Throughout this booklet we will use a variety of symbols for functions and variables to get you used to the fact that our choice of symbols makes no difference to the ideas that we are introducing. On the other hand, we can make life easier for ourselves if we make sensible choices of symbols. For example if we were discussing the revenue obtained by a manufacturer who sells articles for a certain price it might be sensible for us to choose the symbol p to mean price, and r to mean revenue, and to write $r(p)$ to express the fact that the revenue is a function of the price. In this way the symbols we have chosen remind us of their meaning, much more than if we had chosen x to represent price and f to represent revenue and written $f(x)$. On the other hand, because the symbol d has a special use in calculus, to express the derivative $\frac{df(x)}{dx}$, we almost never use d for any other purpose. For this reason you will often see the letter s used to represent di**S**placement.

We now know how to differentiate any function that is a power of the variable. Examples are functions like x^3 and $t^{-1.3}$. You will come across functions that do not at first appear to be a power of the variable, but can be rewritten in this form. One of the simplest examples is the function

$$f(t) = \sqrt{t},$$

which can also be written in the form

$$f(t) = t^{\frac{1}{2}}.$$

The derivative is then

$$f'(t) = \frac{t^{-\frac{1}{2}}}{2} = \frac{1}{2\sqrt{t}}.$$

Similarly, if

$$h(s) = \frac{1}{s} = s^{-1}$$

then

$$h'(s) = -s^{-2} = -\frac{1}{s^2}.$$

Examples

If $f(x) = \dfrac{1}{\sqrt[3]{x}} = x^{-\frac{1}{3}}$ then $f'(x) = -\dfrac{1}{3}x^{-\frac{4}{3}}$.

If $y = \dfrac{1}{x\sqrt{x}} = x^{-\frac{3}{2}}$ then $\dfrac{dy}{dx} = -\dfrac{3}{2}x^{-\frac{5}{2}}$.

Exercises 1.1

Differentiate the following functions:

a. $f(x) = x^4$ **b.** $y = x^{-7}$ **c.** $f(u) = u^{2.3}$

d. $f(t) = t^{-\frac{1}{3}}$ **e.** $f(t) = t^{\frac{22}{7}}$ **f.** $g(z) = z^{-\frac{3}{2}}$

g. $y = t^{-3.8}$ **h.** $z = x^{\frac{3}{7}}$

Exercise 1.2

Express the following as powers and then differentiate:

a. $\dfrac{1}{x^2}$ **b.** $t\sqrt{t}$ **c.** $\sqrt[3]{x}$

d. $\dfrac{1}{x^2\sqrt{x}}$ **e.** $\dfrac{1}{x\sqrt[4]{x}}$ **f.** $\dfrac{s^3\sqrt{s}}{\sqrt[3]{s}}$

g. $\dfrac{1}{u^3}$ **h.** $\dfrac{t}{t^2\sqrt{t}}$ **i.** $x^{\frac{1}{2}}\dfrac{\sqrt{x}}{x}$

Exercise 1.3

Find the equation of the line tangent to the graph of $y = \sqrt[3]{x}$ when $x = 8$.

1.2 Adding, subtracting, and multiplying by a constant

So far we know how to differentiate powers of the independent variable. Many of the functions that you will encounter are made up in simple ways from powers. For example, a function like $3x^2$ is just a constant multiple of x^2. However neither Rule 1 nor Rule 2 tell us how to differentiate $3x^2$. Nor do they tell us how to differentiate something like $x^2 + x^3$ or $x^2 - x^3$.

Rules 3 and 4 specify how to differentiate combinations of functions that are formed by multiplying by constants, or by adding or subtracting functions.

Rule 3 If $f(x) = cg(x)$, where c is a constant, then $f'(x) = cg'(x)$.

Rule 4 If $f(x) = g(x) \pm h(x)$ then $f'(x) = g'(x) \pm h'(x)$.

Examples If $f(x) = 3x^2$ then $f'(x) = 3 \times \dfrac{d}{dx}x^2 = 6x$.

If $g(t) = 3t^2 + 2t^{-2}$ then $g'(t) = \dfrac{d}{dt}3t^2 + \dfrac{d}{dt}2t^{-2} = 6t - 4t^{-3}$.

If $y = \frac{3}{\sqrt{x}} - 2x\sqrt[3]{x} = 3x^{-\frac{1}{2}} - 2x^{\frac{4}{3}}$ then $\frac{dy}{dx} = -\frac{3}{2}x^{-\frac{3}{2}} - \frac{8}{3}x^{\frac{1}{3}}$.

If $y = -0.3x^{-0.4}$ then $\frac{dy}{dx} = 0.12x^{-1.4}$.

$\frac{d}{dx}2x^{0.3} = 0.6x^{-0.7}$.

Warning! Although Rule 4 tells us that $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$, *the same is not true for multiplication or division.* To differentiate $f(x) \times g(x)$ or $f(x) \div g(x)$ we cannot simply find $f'(x)$ and $g'(x)$ and multiply or divide them. *Be careful of this!* The methods of differentiating products of functions or quotients of functions are discussed in Sections 1.3 and 1.4.

Exercise 1.4

Differentiate the following functions:

a. $f(x) = 5x^2 - 2\sqrt{x}$ **b.** $y = 2x^{-7} + \frac{3}{x^2}$ **c.** $f(t) = 2.5t^{2.3} + \frac{t}{\sqrt{t}}$

d. $h(z) = z^{-\frac{1}{3}} + 5z$ e. $f(u) = u^{\frac{5}{3}} - 3u^{-7}$ **f.** $g(z) = 8z^{-2} - \frac{5}{z}$

g. $y = 5t^{-8} + \frac{t}{\sqrt{t}}$ **h.** $z = 4x^{\frac{1}{7}} + 2x^{-\frac{1}{2}}$

1.3 The product rule

Another way of combining functions to make new functions is by multiplying them together, or in other words by forming *products*. The product rule tells us how to differentiate functions like this.

Rule 5 (The product rule) If $f(x) = u(x)v(x)$ then

$$f'(x) = u(x)v'(x) + u'(x)v(x).$$

Examples

If $y = (x+2)(x^2+3)$ then $y' = (x+2)2x + 1(x^2+3)$.

If $f(x) = \sqrt{x}(x^3 - 3x^2 + 7)$ then $f'(x) = \sqrt{x}(3x^2 - 6x) + \frac{1}{2}x^{-\frac{1}{2}}(x^3 - 3x^2 + 7)$.

If $z = (t^2+3)(\sqrt{t} + t^3)$ then $\frac{dz}{dt} = (t^2+3)(\frac{1}{2}t^{-\frac{1}{2}} + 3t^2) + 2t(\sqrt{t} + t^3)$.

Exercise 1.5

Use the product rule to differentiate the functions below:

a. $f(x) = (4x^3 + 2)(1 - 3x)$

b. $g(x) = (x^2 + x + 2)(x^2 + 1)$

c. $h(x) = (3x^3 - 2x^2 + 8x - 5)(x^2 - 2x + 4)$

d. $f(s) = (1 - \frac{1}{2}s^2)(3s + 5)$

e. $g(t) = (\sqrt{t} + \frac{1}{t})(2t - 1)$

f. $h(y) = (2 - \sqrt{y} + y^2)(1 - 3y^2)$

Exercise 1.6

If $r = (t + \frac{1}{t})(t^2 - 2t + 1)$, find the rate of change of r with respect to t when $t = 2$.

Exercise 1.7

Find the slope of the tangent to the curve $y = (x^2 - 2x + 1)(3x^3 - 5x^2 + 2)$ at $x = 2$.

1.4 The Quotient Rule

This rule allows us to differentiate functions which are formed by dividing one function by another, ie by forming *quotients* of functions. An example is such as

$$f(x) = \frac{2x + 3}{3x - 5}.$$

Rule 6 (The quotient rule)

$$\begin{aligned} f(x) &= \frac{u(x)}{v(x)} \\ f'(x) &= \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2} \\ &= \frac{vu' - uv'}{v^2}. \end{aligned}$$

Warning! Because of the minus sign in the numerator (ie in the top line) it is important to get the terms in the numerator in the correct order. This is often a source of mistakes, so be careful. Decide on your own way of remembering the correct order of the terms.

Examples

If $y = \dfrac{2x^2+3x}{x^3+1}$, then $\dfrac{dy}{dx} = \dfrac{(x^3+1)(4x+3)-(2x^2+3x)3x^2}{(x^3+1)^2}$.

If $g(t) = \dfrac{t^2+3t+1}{\sqrt{t}+1}$ then $g'(t) = \dfrac{(\sqrt{t}+1)(2t+3)-(t^2+3t+1)(\frac{1}{2}t^{-\frac{1}{2}})}{(\sqrt{t}+1)^2}$.

Exercise 1.8

Use the Quotient Rule to find derivatives for the following functions:

a. $f(x) = \dfrac{x-1}{x+1}$ b. $g(x) = \dfrac{2x+3}{3x-2}$

c. $h(x) = \dfrac{x^2+2}{x^2+5}$ d. $f(t) = \dfrac{2t}{1+2t^2}$

e. $f(s) = \dfrac{1+\sqrt{s}}{1-\sqrt{s}}$ f. $h(x) = \dfrac{x^2-1}{x^3+4}$

g. $f(u) = \dfrac{u^3+u-4}{3u^4+5}$ h. $g(t) = \dfrac{t(t+6)}{t^2+3t+1}$

1.5 Solutions to exercises

Exercise 1.1

a. $f'(x) = 4x^3$

b. $\dfrac{dy}{dx} = -7x^{-8}$

c. $f'(u) = 2.3u^{1.3}$

d. $f'(t) = -\dfrac{1}{3}t^{-\frac{4}{3}}$

e. $f'(t) = \dfrac{22}{7}t^{\frac{15}{7}}$

f. $g'(z) = -\dfrac{3}{2}z^{-\frac{5}{2}}$

g. $\dfrac{dy}{dt} = -3.8t^{-4.8}$

h. $\dfrac{dz}{dx} = \dfrac{3}{7}x^{-\frac{4}{7}}$

Exercise 1.2

a. $\dfrac{1}{x^2} = x^{-2}$ so $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right) = -2x^{-3}$

b. $t\sqrt{t} = t^{\frac{3}{2}}$ so $\dfrac{d}{dt}\left(t\sqrt{t}\right) = \dfrac{3}{2}t^{\frac{1}{2}}$

c. $\dfrac{d}{dx}\sqrt[3]{x} = \dfrac{d}{dx}x^{\frac{1}{3}} = \dfrac{1}{3}x^{-\frac{2}{3}}$

d. $\dfrac{d}{dx}\left(\dfrac{1}{x^2\sqrt{x}}\right) = \dfrac{d}{dx}x^{-\frac{5}{2}} = -\dfrac{5}{2}x^{-\frac{7}{2}}$

e. $\dfrac{d}{dx}\left(\dfrac{1}{x\sqrt[4]{x}}\right) = -\dfrac{d}{dx}x^{5/4} = \dfrac{-5}{4}x^{-\frac{9}{4}}$

f. $\dfrac{d}{dx}\left(\dfrac{s^3\sqrt{s}}{\sqrt[3]{s}}\right) = \dfrac{d}{ds}s^{\frac{19}{6}} = \dfrac{19}{6}s^{\frac{13}{6}}$

g. $\dfrac{d}{du}\left(\dfrac{1}{u^3}\right) = \dfrac{du^{-3}}{du} = -3u^{-4}$

h. $\dfrac{d}{dt}\left(\dfrac{t}{t^2\sqrt{t}}\right) = \dfrac{d}{dt}t^{-\frac{3}{2}} = -\dfrac{3}{2}t^{-\frac{5}{2}}$

i. $\dfrac{d}{dx}\left(x^{1/2}\dfrac{\sqrt{x}}{x}\right) = \dfrac{d}{dx}1 = 0$

Exercise 1.3

When $x = 8$ we have $y = \sqrt[3]{8} = 2$ so the point $(8, 2)$ is on the line. Now $\frac{dy}{dx} = \frac{x^{-\frac{2}{3}}}{3}$ and so $\frac{dy}{dx} = \frac{1}{12}$ when $x = 8$. The tangent therefore has equation

$$y = \frac{1}{12}x + b.$$

Substituting $x = 8$ and $y = 2$ into this equation we obtain

$$2 = \frac{1}{12}8 + b$$

so that $b = \frac{4}{3}$. The equation is therefore $y = \frac{x}{12} + \frac{4}{3}$.

Exercise 1.4

a. $f'(x) = 10x - x^{-\frac{1}{2}}$ **b.** $\frac{dy}{dx} = -14x^{-8} - 6x^{-3}$ **c.** $f'(t) = 5.75t^{1.3} + \frac{3}{2}t^{\frac{1}{2}}$

d. $h'(z) = -\frac{1}{3}z^{-\frac{4}{3}} + 5$ **e.** $f'(u) = \frac{5}{3}u^{2/3} + 21u^{-8}$ **f.** $g'(z) = -16z^{-3} + 5z^{-2}$

g. $\frac{dy}{dt} = -40t^{-9} + \frac{1}{2}t^{-\frac{1}{2}}$ **h.** $\frac{dz}{dx} = \frac{4}{7}x^{-\frac{6}{7}} - x^{-\frac{3}{2}}$

Exercise 1.5

a. $f'(x) = 12x^2(1-3x) - 3(4x^3+2)$

b. $g'(x) = (2x+1)(x^2+1) + (x^2+x+2)2x$

c. $h'(x) = (9x^2-4x+8)(x^2-2x+4) + (3x^3-2x^2+8x-5)(2x-2)$

d. $f'(s) = -s(3s+5) + 3(1-\frac{s^2}{2})$

e. $g'(t) = (\frac{t^{-\frac{1}{2}}}{2} - t^{-2})(2t-1) + 2(\sqrt{t}+\frac{1}{t})$

f. $h'(y) = (-\frac{y^{-\frac{1}{2}}}{2} + 2y)(1-3y^2) - 6y(2-\sqrt{y}+y^2)$

Exercise 1.6

The rate of change of r with respect to t is

$$\frac{dr}{dt} = (1-t^{-2})(t^2-2t+1) + (t+\frac{1}{t})(2t-2).$$

Substituting $t=2$ we obtain $(1-\frac{1}{4})(4-4+1) + (2+\frac{1}{2})(4-2) = \frac{23}{4}$.

Exercise 1.7

The gradient of the tangent is given by

$$\frac{dy}{dx} = (2x-2)(3x^3-5x^2+2) + (x^2-2x+1)(9x^2-10x).$$

Substituting $x=2$ we obtain 28.

Exercise 1.8

a. $f'(x) = \dfrac{(x+1)-(x-1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$

b. $g'(x) = \dfrac{(3x-2)2-(2x+3)3}{(3x-2)^2} = \dfrac{-13}{(3x-2)^2}$

c. $h'(x) = \dfrac{(x^2+5)2x-(x^2+2)2x}{(x^2+5)^2} = \dfrac{6x}{(x^2+5)^2}$

d. $f'(t) = \dfrac{(1+2t^2)2-8t^2}{(1+2t^2)^2} = \dfrac{2-4t^2}{(1+2t^2)^2}$

e. $f'(s) = \dfrac{(1-\sqrt{s})\frac{1}{2}s^{-\frac{1}{2}} + (1+\sqrt{s})\frac{1}{2}s^{-1/2}}{(1-\sqrt{s})^2} = \dfrac{s^{-\frac{1}{2}}}{(1-\sqrt{s})^2}$

f. $h'(x) = \dfrac{(x^3+4)2x-(x^2-1)3x^2}{(x^3+4)^2} = \dfrac{-x^4+3x^2+8x}{(x^3+4)^2}$

g. $f'(u) = \dfrac{(3u^4+5)(3u^2+1)-(u^3+u-4)12u^3}{(3u^4+5)^2} = \dfrac{-3u^6-9u^4+48u^3+15u^2+5}{(3u^4+5)^2}$

h. $g'(t) = \dfrac{(t^2+3t+1)(2t+6)-(t^2+6t)(2t+3)}{(t^2+3t+1)^2} = \dfrac{-3t^2+2t+6}{(t^2+3t+1)^2}$