Mathematics Learning Centre

The University of Sydney

Solving inequalities

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1 Solving inequalities

In these nots we will look at solving inequalities using both algebraic and graphical techniques.

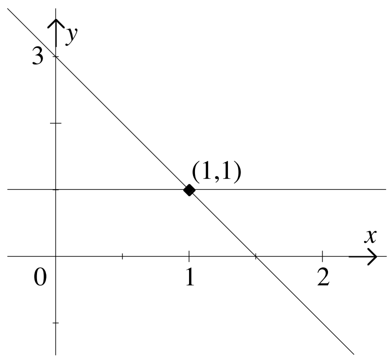
Sometimes it is easier to use an algebraic method and sometimes a graphical one. For the following examples we will use both, as this allows us to make the connections between the algebra and the graphs.

Algebraic method

Graphical method

1. Solve $3 - 2x \geq 1$.
This is a linear inequality. Remember to reverse the inequality sign when multiplying or dividing by a negative number.

$$\begin{aligned} 3 - 2x &\geq 1 \\ -2x &\geq -2 \\ x &\leq 1 \end{aligned}$$

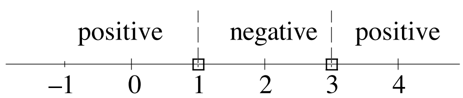


When is the line $y = 3 - 2x$ above or on the horizontal line $y = 1$? From the graph, we see that this is true for $x \leq 1$.

2. Solve $x^2 - 4x + 3 < 0$.
This is a quadratic inequality. Factorise and use a number line.

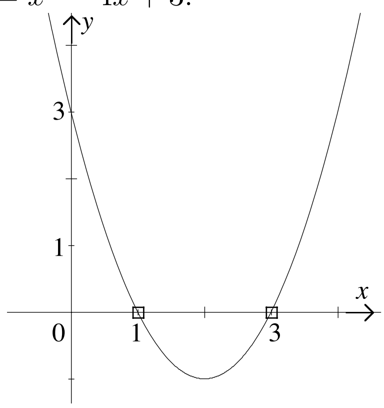
$$\begin{aligned} x^2 - 4x + 3 &< 0 \\ (x-3)(x-1) &< 0 \end{aligned}$$

The critical values are 1 and 3, which divide the number line into three intervals. We take points in each interval to determine the sign of the inequality; eg use $x = 0$, $x = 2$ and $x = 4$ as test values.



Thus, the solution is $1 < x < 3$.

Let $y = x^2 - 4x + 3$.

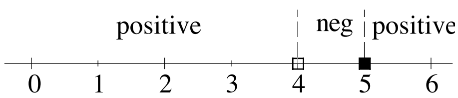


When does the parabola have negative y-values? OR When is the parabola under the x-axis? From the graph, we see that this happens when $1 < x < 3$.

3. Solve $\frac{1}{x-4} \leq 1$.
There is a variable in the denominator. Remember that a denominator can never be zero, so in this case $x \neq 4$.
First multiply by the square of the denominator

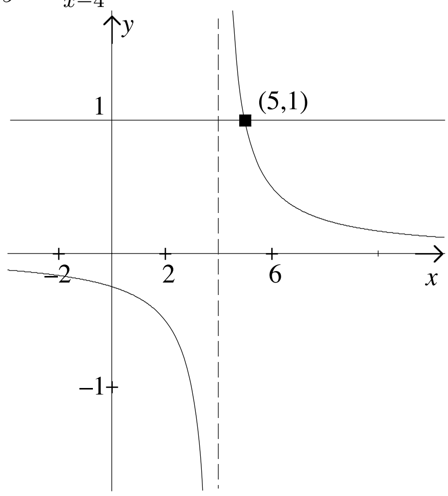
$$\begin{aligned} x-4 &\leq (x-4)^2, x \neq 4 \\ x-4 &\leq x^2-8x+16 \\ 0 &\leq x^2-9x+20 \\ 0 &\leq (x-4)(x-5) \end{aligned}$$

Mark the critical values on the number line and test $x = 0$, $x = 4.5$ and $x = 6$.



Therefore, $x < 4$ or $x \geq 5$.

Let $y = \frac{1}{x-4}$.



$y = \frac{1}{x-4}$ is not defined for $x = 4$. It is a hyperbola with vertical asymptote at $x = 4$. To solve our inequality we need to find the values of x for which the hyperbola lies on or under the line $y = 1$. $(5, 1)$ is the point of intersection. So, from the graph we see that $\frac{1}{x-4} \leq 1$ when $x < 4$ or $x \geq 5$.

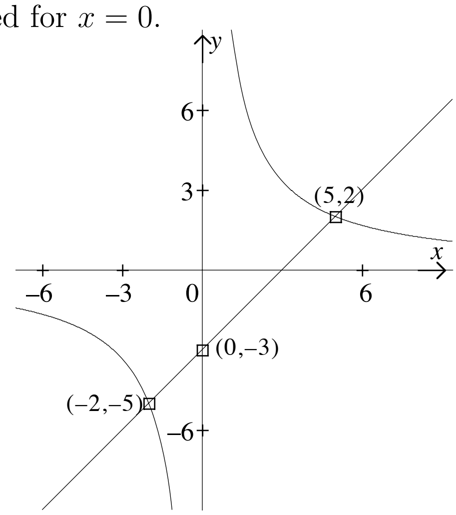
4. Solve $x - 3 < \frac{10}{x}$.
Consider $x - 3 = \frac{10}{x}$, $\quad x \neq 0$.
Multiply by x we get

$$\begin{aligned} x^2-3x &= 10 \\ x^2-3x-10 &= 0 \\ (x-5)(x+2) &= 0 \end{aligned}$$

Therefore, the critical values are -2, 0 and 5 which divide the number line into four intervals. We can use $x = -3$, $x = -1$, $x = 1$ and $x = 6$ as test values in the inequality. The points $x = -3$ and $x = 1$ satisfy the inequality, so the solution is $x < -2$ or $0 < x < 5$.

(Notice that we had to include 0 as one of our critical values.)

Sketch $y = x - 3$ and then $y = \frac{10}{x}$. Note that second of these functions is not defined for $x = 0$.



For what values of x does the line lie under the hyperbola? From the graph, we see that this happens when $x < -2$ or $0 < x < 5$.

Example

Sketch the graph of $y = |2x - 6|$.

Hence, where possible,

a. Solve

i. $|2x - 6| = 2x$

ii. $|2x - 6| > 2x$

iii. $|2x - 6| = x + 3$

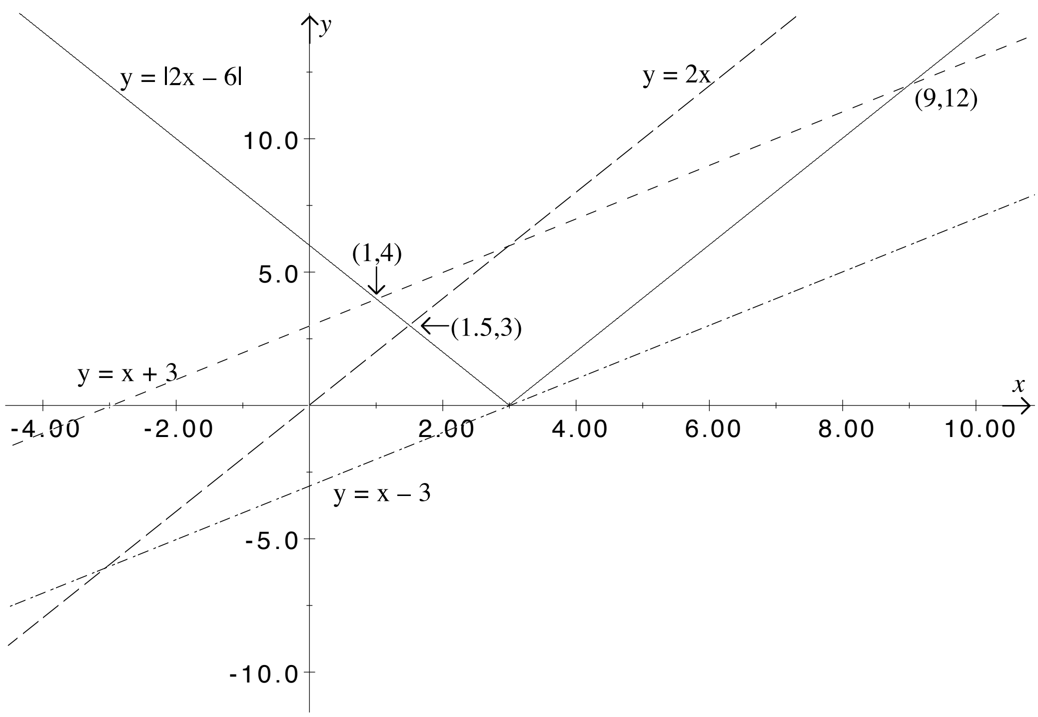
iv. $|2x - 6| < x + 3$

v. $|2x - 6| = x - 3$

b. Determine the values of k for which $|2x - 6| = x + k$ has exactly two solutions.

Solution

$$f(x) = |2x - 6| = \begin{cases} 2x - 6 & \text{for } x \geq 3 \\ -(2x - 6) & \text{for } x < 3 \end{cases}$$



a. i. Mark in the graph of $y = 2x$. It is parallel to one arm of the absolute value graph. It has one point of intersection with $y = |2x - 6| = -2x + 6$ $(x < 3)$ at $x = 1.5$.

ii. When is the absolute value graph above the line $y = 2x$? From the graph, when $x < 1.5$.

iii. $y = x + 3$ intersects $y = |2x - 6|$ twice.
To solve $|2x - 6| = x + 3$, take $|2x - 6| = 2x - 6 = x + 3$ when $x \geq 3$. This gives us the solution $x = 9$. Then take $|2x - 6| = -2x + 6 = x + 3$ when $x < 3$ which gives us the solution $x = 1$.

iv. When is the absolute value graph below the line $y = x + 3$?
From the graph, $1 < x < 9$.

v. $y = x - 3$ intersects the absolute value graph at $x = 3$ only.

b. k represents the y-intercept of the line $y = x + k$. When $k = -3$, there is one point of intersection. (See (a) (v) above). For $k > -3$, lines of the form $y = x + k$ will have two points of intersection. Hence $|2x - 6| = x + k$ will have two solutions for $k > -3$.

1.1 Exercises

1. Solve

a. $x^2 \leq 4x$

b. $\frac{4p}{p+3} \leq 1$

c. $\frac{7}{9-x^2} > -1$

2. **a.** Sketch the graph of $y = 4x(x - 3)$.

b. Hence solve $4x(x - 3) \leq 0$.

3. **a.** Find the points of intersection of the graphs $y = 5 - x$ and $y = \frac{4}{x}$.

b. On the same set of axes, sketch the graphs of $y = 5 - x$ and $y = \frac{4}{x}$.

c. Using part (ii), or otherwise, write down all the values of x for which

$$5 - x > \frac{4}{x}$$

4. **a.** Sketch the graph of $y = 2^x$.

b. Solve $2^x < \frac{1}{2}$.

c. Suppose $0 < a < b$ and consider the points $\mathrm{A}(a, 2^a)$ and $\mathrm{B}(b, 2^b)$ on the graph of $y = 2^x$. Find the coordinates of the midpoint M of the segment AB.
Explain why

$$\frac{2^a + 2^b}{2} > 2^{\frac{a+b}{2}}$$

5. **a.** Sketch the graphs of $y = x$ and $y = |x - 5|$ on the same diagram.

b. Solve $|x - 5| > x$.

c. For what values of m does $mx = |x - 5|$ have exactly

i. two solutions

ii. no solutions

6. Solve $5x^2 - 6x - 3 \leq |8x|$.

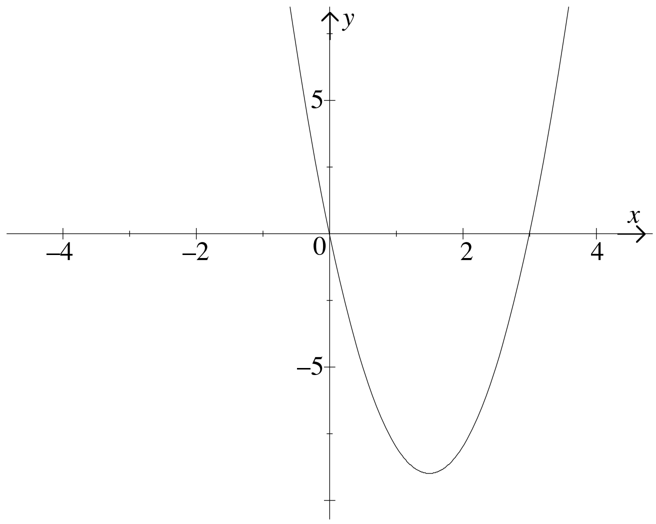
1.2 Solutions

1. **a.** $0 \le x \le 4$

b. $-3 < p \le 1$

c. $x < -4$ or $-3 < x < 3$ or $x > 4$

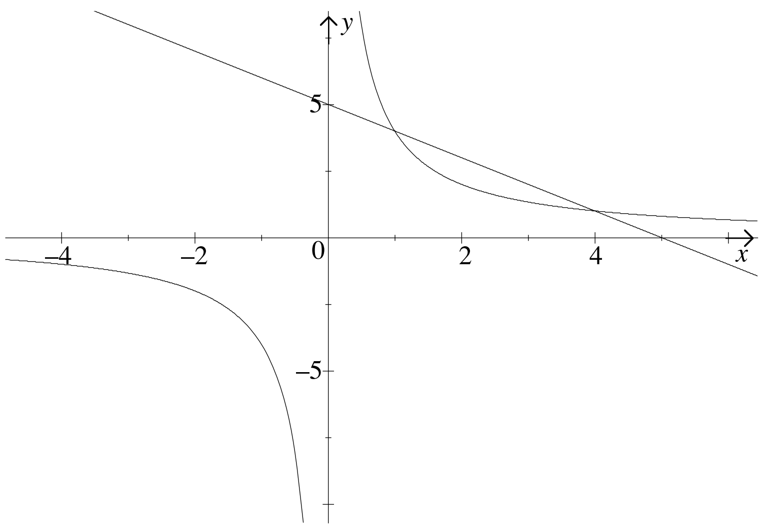
2. **a.** The graph of $y = 4x(x-3)$ is given below



b. From the graph we see that $4x(x-3) \le 0$ when $0 \le x \le 3$.

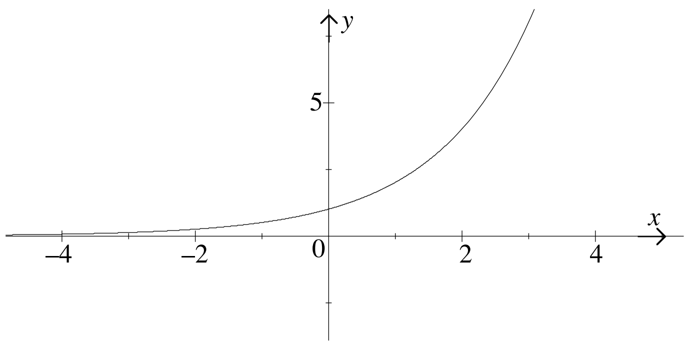
3. **a.** The graphs $y = 5 - x$ and $y = \frac{4}{x}$ intersect at the points $(1, 4)$ and $(4, 1)$.

b. The graphs of $y = 5 - x$ and $y = \frac{4}{x}$



c. The inequality is satisfied for $x < 0$ or $1 < x < 4$.

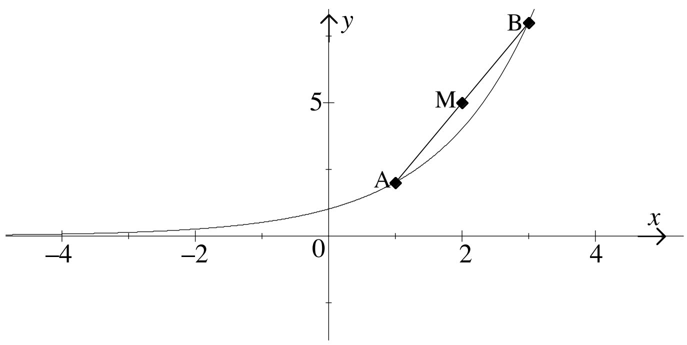
4. **a.** The graph of $y = 2^x$.



b. $2^x < \frac{1}{2}$ when $x < -1$.

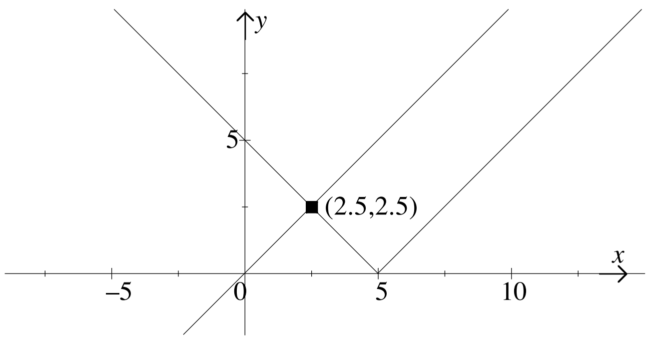
c. The midpoint M of the segment AB has coordinates $(\frac{a+b}{2}, \frac{2^a+2^b}{2})$.
Since the function $y = 2^x$ is concave up, the y-coordinate of M is greater than $f(\frac{a+b}{2})$. So,

$$\frac{2^a + 2^b}{2} > 2^{\frac{a+b}{2}}$$



5.

a.



b. $|x - 5| > x$ for all $x < 2.5$.

c. **i.** $mx = |x - 5|$ has exactly two solutions when $0 < m < 1$.

ii. $mx = |x - 5|$ has no solutions when $-1 < m < 0$.

6. $-1 \leq x \leq 3$