Solutions to Selected Exercises 10

3. ii $y = \sin(x)\cos(x)$

$$\frac{dy}{dx} = \sin x \times \frac{d}{dx}(\cos x) + \cos x \times \frac{d}{dx}(\sin x) = \sin x(-\sin x) + \cos x(\cos x) = \cos^2 x - \sin^2 x.$$

When we differentiate again we can either use the chain rule or treat $\cos^2 x$ and $\sin^2 x$ as products, ie $\cos^2 x = \cos x \cos x$.

Using the product rule we get,

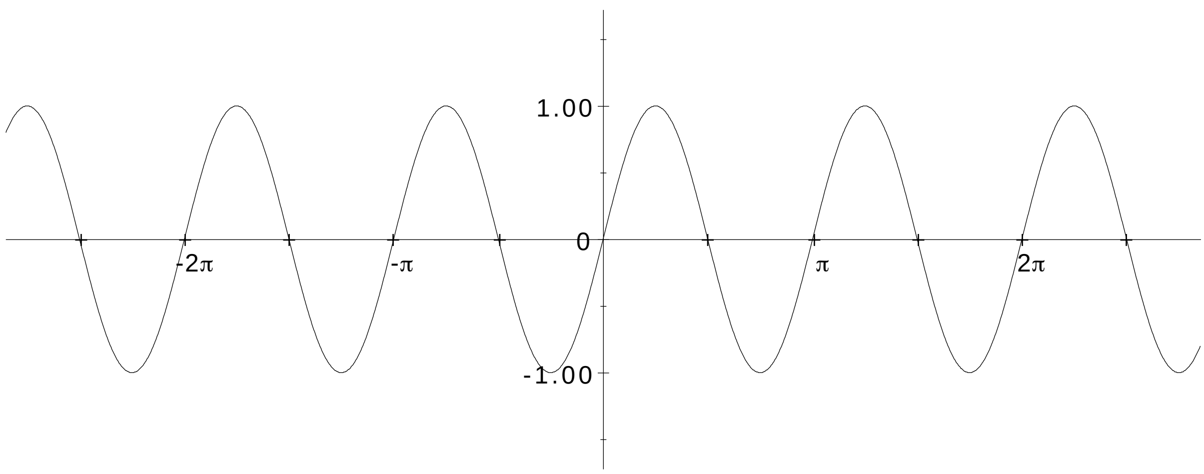
$$\begin{aligned}
\frac{d^2y}{dx^2} &= \cos x \times \frac{d}{dx}(\cos x) + \cos x \times \frac{d}{dx}(\cos x) - \left(\sin x \times \frac{d}{dx}(\sin x) + \sin x \times \frac{d}{dx}(\sin x)\right) \\
&= \cos x(-\sin x) + \cos x(-\sin x) - \sin x(\cos x) - \sin x(\cos x) \\
&= -4\sin x \cos x.
\end{aligned}$$

v $y = x\cos x$

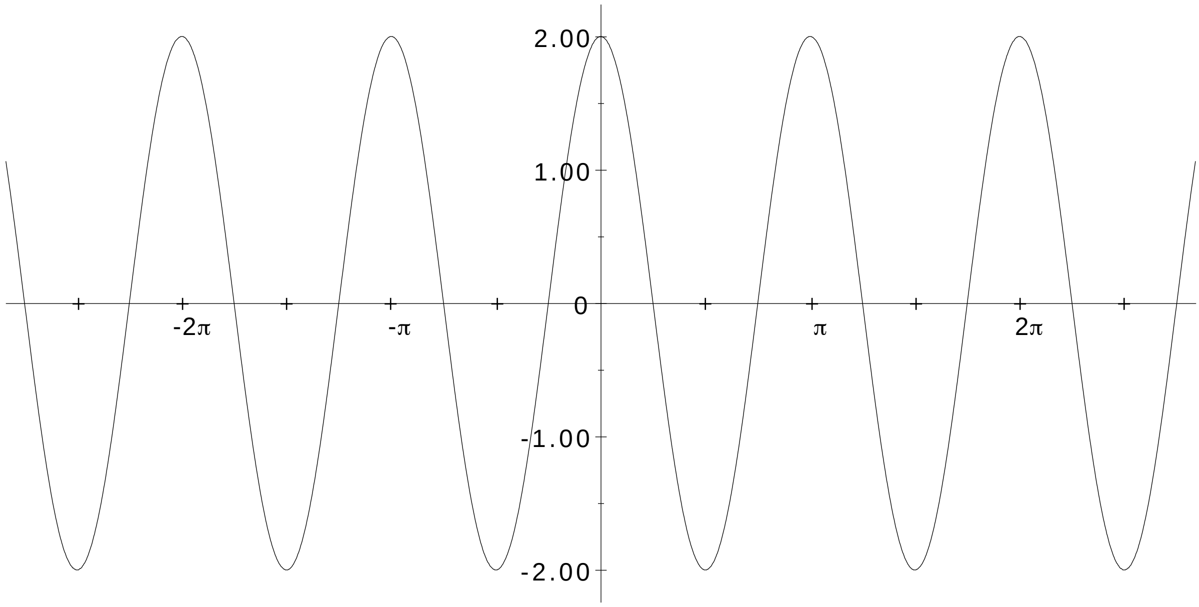
$$\frac{dy}{dx} = x \times \frac{d}{dx}(\cos x) + \cos x \times \frac{d}{dx}(x) = -x\sin x + \cos x.$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= (-x) \times \frac{d}{dx}(\sin x) + \sin x \times \frac{d}{dx}(-x) - \sin x \\
&= (-x)(\cos x) + \sin x(-1) - \sin x \\
&= -x\cos x - 2\sin x.
\end{aligned}$$

4. A sketch of the function $y = \sin 2x$ is given below. Notice that its amplitude is still 1 but its period is π.



This is a sketch of the derivative of $y = \sin 2x$. It is a cos function with amplitude 2, and period π. How close did you get?



5. ii $y = \cos(x + x^2)$

Here $u = f(x) = x + x^2$ and $g(u) = \cos u$, so

$$\frac{dy}{dx} = \underbrace{-\sin(x + x^2)}_{g'(f(x))} \times \frac{d}{dx}(x + x^2) = -(1 + 2x)\sin(x + x^2).$$

iii $y = \sin(x^2)$

Here $u = f(x) = x^2$ and $g(u) = \sin u$, so

$$\frac{dy}{dx} = \underbrace{\cos(x^2)}_{g'(f(x))} \times \frac{d}{dx}(x^2) = 2x\cos(x^2).$$

vi $y = 2\sin(x + \pi)$

Here $u = f(x) = x + \pi$ and $g(u) = 2\sin u$, so

$$\frac{dy}{dx} = \underbrace{2\cos(x + \pi)}_{g'(f(x))} \times \frac{d}{dx}(x + \pi) = 2\cos(x + \pi).$$

Remember that π is a constant so its derivative is zero.

6. **iii** $y = \sin(3x)\cos(x^2)$

$$\begin{aligned}\frac{dy}{dx} &= \sin(3x) \times \frac{d}{dx}(\cos(x^2)) + \cos(x^2) \times \frac{d}{dx}(\sin(3x)) \\ &= \sin(3x)(-\sin(x^2) \times (2x)) + \cos(x^2)(3\cos(3x)) \\ &= -2x\sin(3x)\sin(x^2) + 3\cos(3x)\cos(x^2).\end{aligned}$$

v $y = \dfrac{3x}{1+\cos x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+\cos x) \times \frac{d}{dx}(3x) - 3x \times \frac{d}{dx}(1+\cos x)}{(1+\cos x)^2} \\ &= \frac{3(1+\cos x) - 3x(-\sin x)}{(1+\cos x)^2} \\ &= \frac{3(1+\cos x + x\sin x)}{(1+\cos x)^2}.\end{aligned}$$

vi $y = \dfrac{\sin x}{\cos x} = \tan x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\cos x \times \frac{d}{dx}(\sin x) - \sin x \times \frac{d}{dx}(\cos x)}{(\cos x)^2} \\ &= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} && \sin^2 x + \cos^2 x = 1 \\ &= \frac{1}{\cos^2 x} && \frac{1}{\cos x} = \sec x \\ &= \sec^2 x.\end{aligned}$$

7. **b** $f(x) = \cos(x^2 - 2x + 1)$

$$f'(x) = -\sin(x^2 - 2x + 1) \times \frac{d}{dx}(x^2 - 2x + 1) = -(2x-2)\sin(x^2 - 2x + 1).$$

d $f(x) = \sin(x^{-1})$

$$f'(x) = \cos(x^{-1}) \times \frac{d}{dx}(x^{-1}) = \cos(x^{-1}) \times (-x^{-2}) = -x^{-2}\cos(x^{-1}).$$

e $f(x) = x^2\cos(x^2 + 4)$

$$\begin{aligned}f'(x) &= x^2 \times \frac{d}{dx}(\cos(x^2+4)) + \cos(x^2+4) \times \frac{d}{dx}(x^2) \\ &= x^2(-\sin(x^2+4) \times (2x)) + \cos(x^2+4) \times (2x) \\ &= -2x^3\sin(x^2+4) + 2x\cos(x^2+4).\end{aligned}$$

h $f(x) = \sin(\cos x)$

$$f'(x) = \cos(\cos x) \times \frac{d}{dx}(\cos x) = \cos(\cos x) \times (-\sin x) = -\cos(\cos x) \cdot \sin x.$$

8. ii Write $\cos(x - \pi) = \cos(x + (-\pi))$ and use the identity on page 37 of the notes.

$$\cos(x - \pi) = \cos(x + (-\pi)) = \cos x \cos(-\pi) - \sin x \sin(-\pi) = -\cos x.$$

Note that $\cos(-\pi) = \cos \pi = -1$ and $\sin(-\pi) = -\sin \pi = 0$.

iv Write $\cos(2\pi - x) = \cos(2\pi + (-x))$.

$$\cos(2\pi - x) = \cos 2\pi \cos(-x) - \sin 2\pi \sin(-x) = \cos(-x) = \cos x.$$

Note that $\cos 2\pi = 1$, $\sin 2\pi = 0$ and $\cos(-x) = \cos x$.