Solutions to Selected Exercises 12

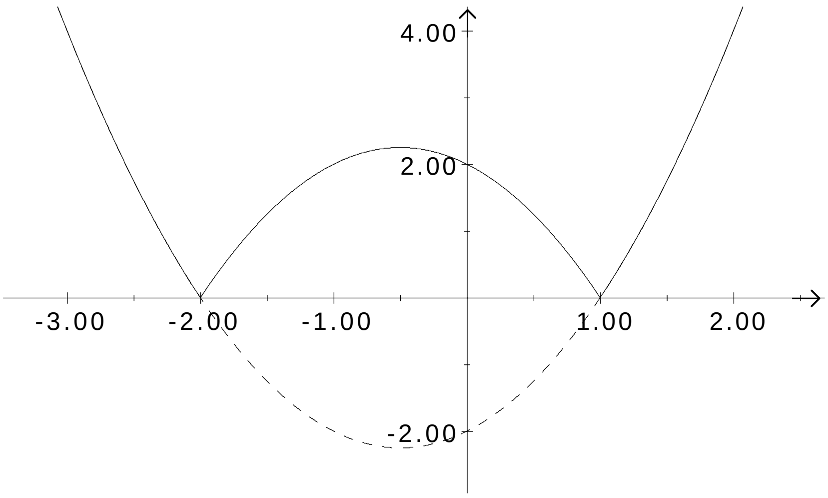
1.

$$||2| - |-5|| = |2-5| = |-3| = 3.$$

2. **iii** If $|2+x| = 5$ then $2+x = 5$ or $2+x = -5$. So, $x = 3$ or $x = -7$.

3. **ii** $y = |x^2 + x - 2|$

The easiest way to draw this graph is to sketch the graph of the function $y = x^2+x-2$ and reflect the part below the x-axis in the x-axis.



4. **i**

The gradient of the line is $m = \dfrac{4-2}{-1-1} = -1$.

Let $y = -x + b$. When $x = 1$ $y = 2$ so, $2 = -1(1) + b$ ie $b = 3$.

The equation of the line is $y = -x + 3$ or $x + y - 3 = 0$.

5. **ii** To find the point(s) of intersection of the two curves we write the equation of the line as $y = x - 1$, let $\dfrac{1}{x-1} = x - 1$ and solve for x.

$$\begin{aligned}
\frac{1}{x-1} &= x-1 \\
(x-1)^2 &= 1 \\
x^2 - 2x + 1 &= 1 \\
x^2 - 2x &= 0 \\
x(x-2) &= 0
\end{aligned}$$

Therefore, $x = 0$ or $x = 2$. Substitute to find y.

The two curves intersect at $(0, -1)$ and $(2, 1)$.

7. Let the length of the sides of the rectangle be x and y. Then the perimeter of the rectangle is $2x + 2y = 40$ and the area, A, is $A = xy = 96$.

Substituting $y = 20 - x$ in A we get,

$$A = x(20 - x) = 20x - x^2 = 96 \quad \text{or} \quad x^2 - 20x + 96 = 0.$$

That is, $x^2 - 20x + 96 = (x - 8)(x - 12) = 0$, ie $x = 8$ or $x = 12$.

Therefore the dimensions of the rectangle are 8 cms by 12 cms.

8. **a**

$$v^3\sqrt{\frac{1}{v}} = v^3\left(\frac{1}{v}\right)^{\frac{1}{2}} = v^3(v^{-1})^{\frac{1}{2}} = v^3(v^{-\frac{1}{2}}) = v^{\frac{5}{2}}.$$

d

$$\sqrt[3]{\frac{x^7}{x^3}} = \left(\frac{x^7}{x^3}\right)^{\frac{1}{3}} = \frac{x^{\frac{7}{3}}}{x^{\frac{3}{3}}} = x^{\frac{7}{3}}x^{-1} = x^{\frac{4}{3}}.$$

h

$$\frac{v^4}{v^{\frac{3}{2}}v^{-2}} = v^4v^{-\frac{3}{2}}v^{-(-2)} = v^{4-\frac{3}{2}+2} = v^{\frac{9}{2}}.$$

l

$$\frac{\sqrt[4]{x}}{x^2} = \frac{x^{\frac{1}{4}}}{x^2} = x^{\frac{1}{4}}x^{-2} = x^{-\frac{7}{4}}.$$

10. **iv**

$$\frac{d}{dx}(x\sin x) = x(\cos x) + \sin x(1) = x\cos x + \sin x.$$

viii Let $u = f(x) = \sin x$ and $g(u) = e^u$ so,

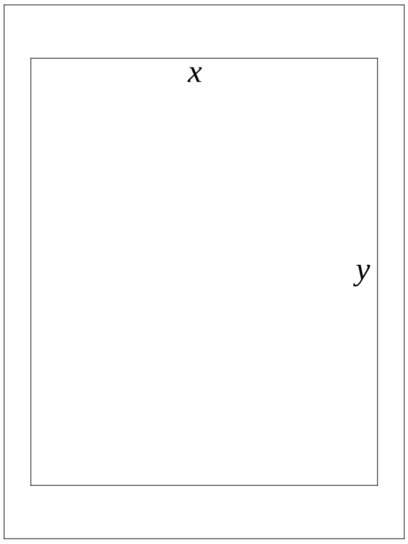
$$\frac{d}{dx}(e^{\sin x}) = \underbrace{e^{\sin x}}_{g'(f(x))} \times \frac{d}{dx}(\sin x) = e^{\sin x}\cos x.$$

ix Let $u = f(x) = \frac{1}{x}$ and $g(u) = \tan u$ so,

$$\frac{d}{dx}(\tan\frac{1}{x}) = \underbrace{\sec^2\frac{1}{x}}_{g'(f(x))} \times \frac{d}{dx}(x^{-1}) = -\frac{\sec^2\frac{1}{x}}{x^2}.$$

Note that $\dfrac{d}{du}(\tan u) = \sec^2 u$.

11. Let the internal dimensions of the advertisement be x and y as shown in the diagram.



Then the overall dimensions are $x+4$ and $y+8$. We also know that the internal area is 50 cm^2, ie $xy = 50$.

Let A be the overall area of the advertisement so

$A = (x+4)(y+8) = (x+4)\left(\frac{50}{x}+8\right) = (x+4)\left(\frac{50+8x}{x}\right) = \frac{8x^2+82x+200}{x}$.

$$\begin{aligned}\frac{dA}{dx} &= (8x^2+82x+200)\times(-x^{-2}) + (x^{-1})\times(16x+82) \\ &= -\frac{8x^2+82x+200}{x^2} + \frac{16x+82}{x} \\ &= \frac{16x^2+82x-(8x^2+82x+200)}{x^2} \\ &= \frac{8x^2-200}{x^2}.\end{aligned}$$

$\frac{dA}{dx} = 0$ when $8x^2 = 200$ ie when $x = \pm 5$. Clearly $x = -5$ cannot be a solution.
The table below tells us that we have a minimum when $x = 5$.

x	< 5	5	> 5
A'	$-ve$	0	$+ve$
A	$\searrow$	162	$\nearrow$

When $x = 5$, $y = 10$ and the overall area is 162 cm^2, so the overall dimensions that make the area of the advertisement a minimum are 9 cms by 18 cms.

14. ii

The function $y = x^2e^x$ is positive for all values of $x \neq 0$, so the graph of the function is above the x-axis for all values of $x \neq 0$ and touches it at $(0, 0)$. Differentiating we get,

$$\frac{dy}{dx} = x^2e^x + e^x(2x) = xe^x(x + 2).$$

This is equal to zero when $x = 0$ or $x = -2$, and so the stationary points are $(-2, 4e^{-2})$ and $(0, 0)$.

x	< -2	-2	$> -2, < 0$	0	> 0
y'	$+ve$	0	$-ve$	0	$+ve$
y	$\nearrow$	$4e^{-2}$	$\searrow$	0	$\nearrow$

The table indicates that we have a maximum at $(-2, 4e^{-2})$ and a minimum at $(0, 0)$.

$$\frac{d^2y}{dx^2} = x^2e^x + e^x(2x) + 2xe^x + e^x(2) = e^x(x^2 + 4x + 2).$$

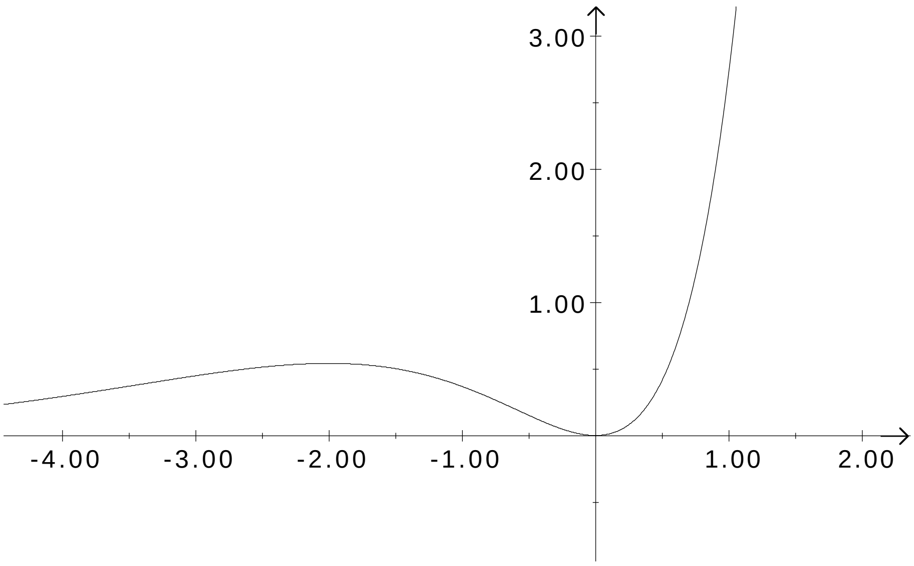
$\dfrac{d^2y}{dx^2} = 0$ when $x^2 + 4x + 2 = 0$. We solve this using the quadratic formula.

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{(4)^2 - 4(1)(2)}}{2} \\ &= \frac{-4 \pm \sqrt{8}}{2} \\ &= \frac{-4 \pm 2\sqrt{2}}{2} \\ &= -2 \pm \sqrt{2}. \end{aligned}$$

We have possible inflection points when $x = -2 - \sqrt{2}$ and $x = -2 + \sqrt{2}$, which we confirm in the following table.

x	$< -2 - \sqrt{2}$	$-2 - \sqrt{2}$	$> -2 - \sqrt{2}, < -2 + \sqrt{2}$	$-2 + \sqrt{2}$	$> -2 + \sqrt{2}$
y''	$+ve$	0	$-ve$	0	$+ve$
y	concave up	0.38	concave down	0.19	concave up

We now have enough information to sketch the curve.



16. ii

$$\begin{aligned}\log\left(\frac{10^x}{100^x}\right) &= \log(10^x) - \log(100^x)\\ &= x\log 10 - x\log 100\\ &= x - x\log(10^2)\\ &= x - 2x\log 10\\ &= x - 2x\\ &= -x.\end{aligned}$$

17. ii

$$\begin{aligned}\ln(x^2+1) &= 3\\ e^{\ln(x^2+1)} &= e^3\\ x^2+1 &= e^3\\ x^2 &= e^3-1\\ x &= \pm\sqrt{e^3-1}\\ &= \pm 4.37.\end{aligned}$$

Both of these solutions are valid as $(\pm 4.37)^2 + 1 > 0$.

19. The population at $t = 0$ is $P = Ae^0 = A$. The population doubles after 5 years so when $t = 5$, $P = 2A$.

i When $t = 5$, $P = 2A$ so, $2A = Ae^{5k}$ ie $e^{5k} = 2$. Therefore, $k = \dfrac{\ln 2}{5}$.

ii When $t = 20$ $P = Ae^{\frac{\ln 2}{5}\cdot 20} = Ae^{4\ln 2} = Ae^{\ln(2^4)} = 16A$.

iii We need to find the value of t when $P = 4A$. So

$$4A = Ae^{\frac{\ln 2}{5}t}$$

$$\begin{aligned} e^{\frac{\ln 2}{5}t} &= 4 \\ \frac{\ln 2}{5}t &= \ln 4 \\ t &= \frac{5\ln 4}{\ln 2} \\ &= 10. \end{aligned}$$

Therefore, the population is four times its initial value after 10 years.

21. ii $\sin u = 1$ when $u = -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$ etc.

$\sin\left(3x - \dfrac{\pi}{2}\right) = 1$ when $\left(3x - \dfrac{\pi}{2}\right) = -\dfrac{3\pi}{2}$ or $\left(3x - \dfrac{\pi}{2}\right) = \dfrac{\pi}{2}$ or $\left(3x - \dfrac{\pi}{2}\right) = \dfrac{5\pi}{2}$ or $\left(3x - \dfrac{\pi}{2}\right) = \dfrac{7\pi}{2}$ etc.

That is, $\sin\left(3x - \dfrac{\pi}{2}\right) = 1$ when $x = -\frac{\pi}{3}$ or $x = \frac{\pi}{3}$ or $x = \pi$ or $x = \frac{5\pi}{3}$ etc.

So, the values of x between 0 and 2π for which $\sin\left(3x - \dfrac{\pi}{2}\right) = 1$ are $x = \frac{\pi}{3}$, $x = \pi$ and $x = \frac{5\pi}{3}$.

23. i When $d = 300$, $I = 0.3I(0)$ so,

$$\begin{aligned} 0.3I(0) &= I(0)e^{-300k} \\ e^{-300k} &= 0.3 \\ -300k &= \ln 0.3 \\ k &= -\frac{\ln 0.3}{300} \\ &= 0.004. \end{aligned}$$

ii We want to find d when $I(d) = 0.5I(0)$ so,

$$\begin{aligned} 0.5I(0) &= I(0)e^{-0.004d} \\ e^{-0.004d} &= 0.5 \\ -0.004d &= \ln 0.5 \\ d &= -\frac{\ln 0.5}{0.004} \\ &= 173.3. \end{aligned}$$

Therefore, the intensity of the sunlight would be decreased by half at a depth of 173 units below the surface.