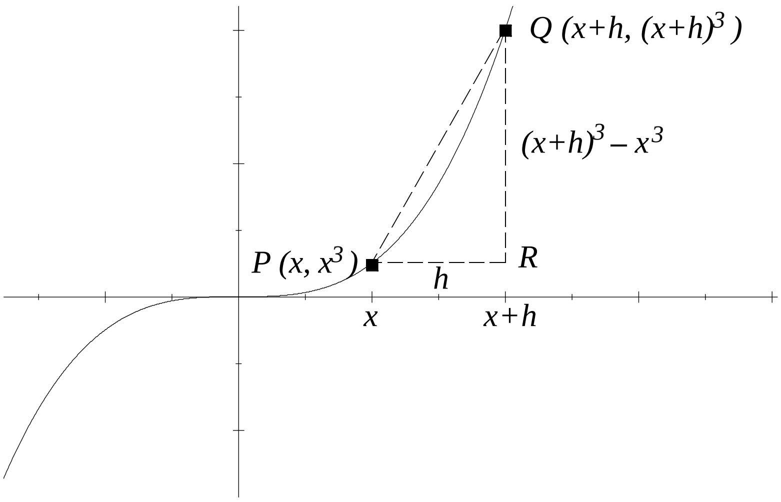
Solutions to Selected Exercises 5

1.

i Let P be the point (x, x^3) and Q be the point $(x+h, (x+h)^3)$ on the curve $y = x^3$. Let R be the point $(x+h, x^3)$.



The gradient of the chord PQ is given by

$$\begin{aligned}
\text{gradient of } PQ &= \frac{QR}{PR} \\
&= \frac{(x+h)^3 - x^3}{x+h-x} \\
&= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= 3x^2 + 3xh + h^2.
\end{aligned}$$

ii As $h \to 0$, $3xh + h^2 \to 0$ so the gradient of $PQ \to 3x^2$.

iii

$$\frac{d}{dx}(x^3) = \lim_{h \to 0} \text{ gradient of } PQ = 3x^2.$$

3. b $f'(x) = 12x^3 + 4x - 1$ so $f'(0) = -1$.

5. Before we differentiate to find any stationary points we will determine where the graph crosses the axes.

When $x = 0$, $y = 0^3(0 - 2) = 0$ so the graph cuts the y-axis at 0.

When $y = 0$, $0 = x^3(x - 2)$ so $x = 0$ or $x = 2$. The graph cuts the x-axis at 0 and 2.

Now $y = x^3(x - 2) = x^4 - 2x^3$, so $y' = 4x^3 - 6x^2$.

Now, $4x^3 - 6x^2 = 2x^2(2x - 3) = 0$ when $x = 0$ or $x = \frac{3}{2}$ so there are two stationary points: $(0, 0)$ and $(\frac{3}{2}, -\frac{27}{16})$.

It is useful to draw up a table as follows to determine the nature of the stationary points.

x	< 0	0	$> 0, \quad < \frac{3}{2}$	$\frac{3}{2}$	$> \frac{3}{2}$
y'	$-ve$	0	$-ve$	0	$+ve$
y	$\searrow$	0	$\searrow$	$-\frac{27}{16}$	$\nearrow$

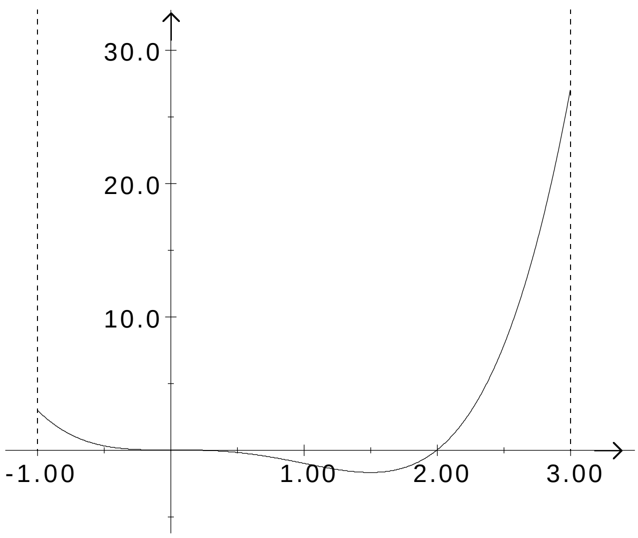
The table tells us that the function is decreasing for $x < 0$ and decreasing in the interval $0 < x < \frac{3}{2}$. Therefore the stationary point at $(0, 0)$ is neither a maximum nor a minimum.

As the function is decreasing in the interval $0 < x < \frac{3}{2}$ and increasing for $x > \frac{3}{2}$, the stationary point at $(\frac{3}{2}, -\frac{27}{16})$ is a minimum.

To complete the picture we need the values of y for $x = -1$ and $x = 3$.

When $x = -1$, $y = (-1)^3(-1 - 2) = (-1)(-3) = 3$. When $x = 3$, $y = (3)^3(3 - 2) = 27$.

Putting all this information together we can now sketch the curve.



7.

i When $x = 1$, $y = (1)^2 + 5(1) - 8 = -2$.

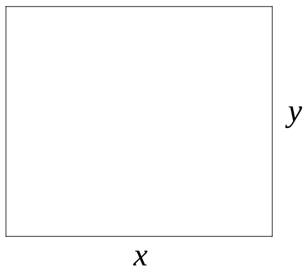
ii $y' = 2x + 5$, so when $x = 1$, $y' = 2(1) + 5 = 7$.
So the value of the derivative at the point $(1, -2)$ is 7.

iii The tangent to the curve at $(1, -2)$ has gradient 7 and passes through $(1, -2)$.

Let $y = 7x + b$. When $x = 1$, $y = -2$ so $-2 = 7(1) + b$ ie $b = -9$.

The equation of the tangent to the curve $y = x^2 + 5x - 7$ at the point $(1, -2)$ is $y = 7x - 9$ or $7x - y - 9 = 0$.

9. Let the perimeter of the rectangle be $2p$ where p is a constant. Let the sides of the rectangle be of length x and y.



Then $2x + 2y = 2p$ ie $y = p - x$. (Making the perimeter $2p$ instead of p makes the algebra a bit easier.)

The area of the rectangle A is given by $A = xy = x(p - x) = px - x^2$.

To maximise the area we need to differentiate A with respect to x and set the derivative equal to 0.
That is, $A' = p - 2x = 0$ (remember that p is a constant).

When $A' = 0$, $p - 2x = 0$ ie $x = \frac{p}{2}$. We know we have a maximum when $x = \frac{p}{2}$ as the coefficient of the x^2 term in $A = px - x^2$ is negative.

When $x = \frac{p}{2}$, $y = p - \frac{p}{2} = \frac{p}{2}$ so the rectangle with the maximum area for a given perimeter is a square.