Solutions to Selected Exercises 6

2. iii $y = x^3 - 12x + 12$

$y' = 3x^2 - 12$. There are stationary points when $y' = 0$,

$$y' = 3x^2 - 12 = 3(x^2 - 4) = 3(x-2)(x+2) = 0 \quad \text{ie when} \quad x = \pm 2.$$

So, $(-2, 28)$ and $(2, -4)$ are stationary points.

We will investigate the nature of the stationary points by drawing up a table.

x	< -2	-2	$> -2, \quad < 2$	2	> 2
y'	$+ve$	0	$-ve$	0	$+ve$
y	$\nearrow$	28	$\searrow$	-4	$\nearrow$

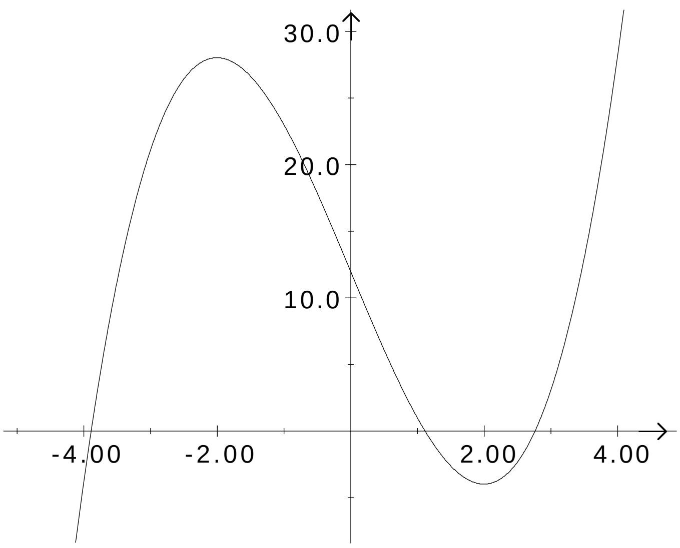
The table tells us that the function is increasing for $x < -2$ and decreasing for $-2 < x < 2$ so the point $(-2, 28)$ is a maximum. The function is decreasing for $-2 < x < 2$ and increasing for $x > 2$, so the point $(2, -4)$ is a minimum.

A point of inflection occurs when $\frac{d^2y}{dx^2} = 0$ and there is a change of concavity.

$\frac{d^2y}{dx^2} = 6x = 0$ ie $x = 0$.

x	< 0	0	> 0
y''	$-ve$	0	$+ve$
y	concave down	$+12$	concave up

From the table we see that when $x = 0$, $\frac{d^2y}{dx^2} = 0$ and there is a change of concavity from concave down to concave up. Therefore, there is a point of inflection at $(0, 12)$.



vi $\quad y = (x-1)^3$

We must expand out the bracket in order to differentiate.

$$y = (x-1)^3 = (x-1)(x^2-2x+1) = x^3 - 2x^2 + x - x^2 + 2x - 1 = x^3 - 3x^2 + 3x - 1.$$

Differentiating and setting the derivative equal to 0, we get

$$\frac{dy}{dx} = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2 = 0 \quad \text{ie} \quad x = 1.$$

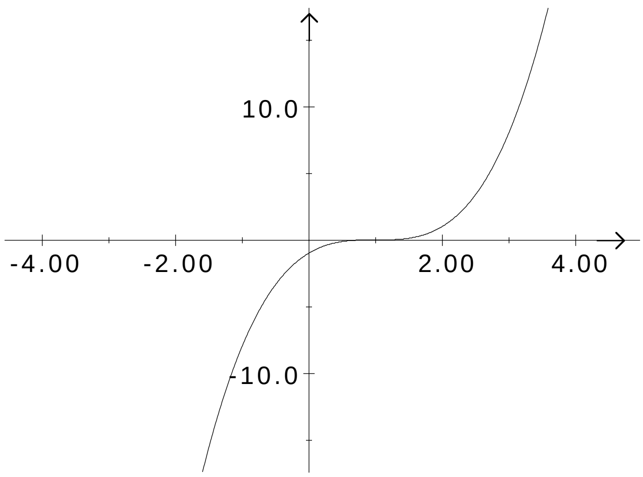
x	< 1	1	> 1
y'	$+ve$	0	$+ve$
y	$\nearrow$	0	$\nearrow$

Since the function is increasing for $x < 1$ and increasing for $x > 1$, the point $(1, 0)$ is a (horizontal) point of inflection.

Differentiating again to find the second derivative we get,

$$\frac{d^2y}{dx^2} = 6x - 6 = 6(x-1).$$

This is equal to 0 when $x = 1$ as before, so there are no other points of inflection.



vii $\quad y = x^4 - 2x^2$

$y' = 4x^3 - 4x = 4x(x^2-1) = 0$ when $x = 0$ or $x \pm 1$.

x	< -1	-1	$> -1, < 0$	0	$> 0, < 1$	1	> 1
y'	$-ve$	0	$+ve$	0	$-ve$	0	$+ve$
y	$\searrow$	-1	$\nearrow$	0	$\searrow$	-1	$\nearrow$

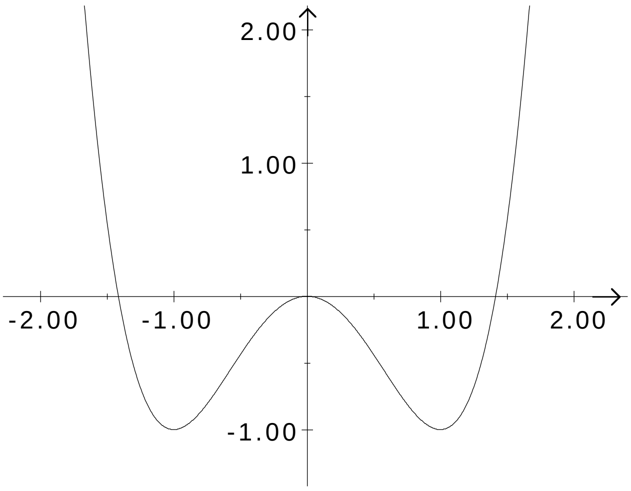
As the function is decreasing for $x < -1$ and increasing for $-1 < x < 0$, the point $(-1,-1)$ is a minimum. As the function is increasing for $-1 < x < 0$ and decreasing for $0 < x < 1$, the point $(0,0)$ is a maximum. As the function is decreasing for $0 < x < 1$ and increasing for $x > 1$, the point $(1,-1)$ is a minimum.

Points of inflection occur when $y'' = 0$ and the concavity changes.

$y'' = 12x^2 - 4 = 4(3x^2 - 1) = 0$ ie when $x = \pm\frac{1}{\sqrt{3}}$.

x	$< -\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$> -\frac{1}{\sqrt{3}}, \quad < \frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$> \frac{1}{\sqrt{3}}$
y''	$+ve$	0	$-ve$	0	$+ve$
y	concave up	$-\frac{5}{9}$	concave down	$-\frac{5}{9}$	concave up

We see from the table that $(-\frac{1}{\sqrt{3}}, -\frac{5}{9})$ and $(\frac{1}{\sqrt{3}}, -\frac{5}{9})$ are points of inflection.



3.

i The velocity at time t is given by

$$\frac{ds}{dt} = 24 - 9.8t.$$

When $t = 0$, $\frac{ds}{dt} = 24$, so the initial velocity of the rock is 24 m/sec.

ii The rock achieves its maximum height when the velocity is zero. $\frac{ds}{dt} = 0$ when $t = \frac{24}{9.8} = 2.45$ sec.

The maximum height of the rock is $s = 24(2.45) - 4.9(2.45)^2 = 29.4$ metres.

iii When the rock hits the ground $s = 0$, so $24t - 4.9t^2 = t(24 - 4.9t) = 0$ ie $t = 0$ or $t = \frac{24}{4.9} = 4.90$.

The rock takes about 5 seconds to fall back to the ground.

5. The velocity is zero when $\frac{ds}{dt} = 3t^2 - 8t - 3 = (3t+1)(t-3) = 0$ ie when $t = 3$. (We can discard $t = -\frac{1}{3}$ as it is negative.)

The acceleration of the body is given by

$$\frac{d^2s}{dx^2} = 6t - 8.$$

When $t = 3$, $\frac{d^2s}{dx^2} = 6(3) - 8 = 10$.

So the acceleration of the body is 10 m/sec^2 when the velocity of the body is zero.