Mathematics Learning Centre

The University of Sydney

Integration: The anti-derivative

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1 Definition of the integral as an anti-derivative

If $\dfrac{d}{dx}(F(x)) = f(x)$ then $\int f(x)dx = F(x)$.

In words,

> If the derivative of $F(x)$ is $f(x)$, then we say that an indefinite integral of $f(x)$ with respect to x is $F(x)$.

For example, since the derivative (with respect to x) of x^2 is $2x$, we can say that an indefinite integral of $2x$ is x^2.

In symbols:

$$\frac{d}{dx}(x^2) = 2x, \qquad \text{so} \qquad \int 2x dx = x^2.$$

Note that we say *an* indefinite integral, not *the* indefinite integral. This is because the indefinite integral is not unique. In our example, notice that the derivative of $x^2 + 3$ is also $2x$, so $x^2 + 3$ is another indefinite integral of $2x$. In fact, if c is *any* constant, the derivative of $x^2 + c$ is $2x$ and so $x^2 + c$ is an indefinite integral of $2x$.

We express this in symbols by writing

$$\int 2x dx = x^2 + c$$

where c is what we call an "arbitrary constant". This means that c has no specified value, but can be given any value we like in a particular problem. In this way we encapsulate all possible solutions to the problem of finding an indefinite integral of $2x$ in a single expression.

In most cases, if you are asked to find an indefinite integral of a function, it is not necessary to add the $+c$. However, there are cases in which it is essential. For example, if additional information is given and a specific function has to be found, or if the general solution of a differential equation is sought. (You will learn about these later.) So it is a good idea to get into the habit of adding the arbitrary constant every time, so that when it is really needed you don't forget it.

The inverse relationship between differentiation and integration means that, for every statement about differentiation, we can write down a corresponding statement about integration.

For example,

$$\frac{d}{dx}(x^4) = 4x^3, \qquad \text{so} \qquad \int 4x^3 dx = x^4 + c.$$

Exercises 1.1

Complete the following statements:

(i) $\frac{d}{dx}(\sin x) = \cos x$, so $\int \cos x \, dx = \sin x + c.$

(ii) $\frac{d}{dx}(x^5) =$, so $\int \quad dx =$

(iii) $\frac{d}{dx}(e^x) =$, so $\int \quad dx =$

(iv) $\frac{d}{dx}\left(\frac{1}{x^2}\right) =$, so $\int \quad dx =$

(v) $\frac{d}{dx}(x) =$, so $\int \quad dx =$

(vi) $\frac{d}{dx}(\ln x) =$, so $\int \quad dx =$

The next step is, when we are given a function to integrate, to run quickly through all the standard differentiation formulae in our minds, until we come to one which *fits* our problem.

In other words, we have to learn to recognise a given function as the derivative of another function (where possible).

Try to do the following exercises by *recognising the function* which has the given function as its derivative.

Exercises 1.2

i. $\int (-\sin x) dx$

ii. $\int 3x^2 dx$

iii. $\int 2 dx$

iv. $\int \sec^2 x dx$

v. $\int \frac{3}{2} x^{\frac{1}{2}} dx$

vi. $\int \left(-\frac{1}{x^2}\right) dx$

2 Some rules for calculating integrals

Rules for operating with integrals are derived from the rules for operating with derivatives. So, because

$$\frac{d}{dx}\left(cf(x)\right) = c\frac{d}{dx}\left(f(x)\right), \text{ for any constant } c,$$

we have

Rule 1

$$\int (cf(x))dx = c\int f(x)dx, \text{ for any constant } c.$$

For example $\int 10\cos x dx = 10\int \cos x dx = 10\sin x + c.$

It sometimes helps people to understand and remember rules like this if they say them in words. The rule given above says: *The integral of a constant multiple of a function is a constant multiple of the integral of the function.* Another way of putting it is *You can move a constant past the integral sign without changing the value of the expression.*

Similarly, from

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)),$$

we can derive the rule

Rule 2

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx.$$

For example,
$$\begin{aligned}\int (e^x + 2x)dx &= \int e^x dx + \int 2x dx \\ &= e^x + x^2 + c.\end{aligned}$$

In words, *the integral of the sum of two functions is the sum of their integrals.*

We can easily extend this rule to include differences as well as sums, and to the case where there are more than two terms in the sum.

Examples

Find the following indefinite integrals:

i. $\int (1 + 2x - 3x^2 + \sin x)dx$

ii. $\int (3\cos x - \frac{1}{2}e^x)dx$

Solutions

i.

$$\begin{aligned}\int (1 + 2x - 3x^2 + \sin x)dx &= \int 1dx + \int 2xdx - \int 3x^2dx - \int(-\sin x)dx \\ &= x + x^2 - x^3 - \cos x + c.\end{aligned}$$

Note: We have written $+\int \sin x dx$ as $-\int(-\sin x)dx$ because $(-\sin x)$ is the derivative of $\cos x$.

ii.

$$\begin{aligned}\int (3\cos x - \frac{1}{2}e^x)dx &= 3\int \cos x dx - \frac{1}{2}\int e^x dx \\ &= 3\sin x - \frac{1}{2}e^x + c.\end{aligned}$$

You will find you can usually omit the first step and write the answer immediately.

Exercises 2

Find the following indefinite integrals:

i. $\int (\cos x + \sin x)dx$

ii. $\int (e^x - 1)dx$

iii. $\int (1 - 10x + 9x^2)dx$

iv. $\int (3\sec^2 x + \frac{4}{x})dx$

3 Integrating powers of x and other elementary functions

We can now work out how to integrate any power of x by looking at the corresponding rule for differentiation:

$$\frac{d}{dx}(x^n) = nx^{n-1}, \qquad \text{so} \qquad \int nx^{n-1}dx = x^n + c.$$

Similarly

$$\frac{d}{dx}\left(x^{n+1}\right) = (n+1)x^n, \qquad \text{so} \qquad \int (n+1)x^n dx = x^{n+1} + c.$$

Therefore

$$\begin{aligned} \int x^n dx &= \int \frac{1}{n+1} \cdot (n+1)x^n dx && \leftarrow \text{notice that } \tfrac{1}{n+1} \cdot (n+1) \text{ is just 1 when we cancel} \\ &= \frac{1}{n+1} \int (n+1)x^n dx && \leftarrow \text{take } \tfrac{1}{n+1} \text{ outside the } \textstyle\int \text{ sign} \\ &= \frac{1}{n+1} x^{n+1} + c. \end{aligned}$$

We should now look carefully at the formula we have just worked out and ask: for which values of n does it hold?

Remember that the differentiation rule $\frac{d}{dx}(x^n) = nx^{n-1}$ holds whether n is positive or negative, a whole number or a fraction or even irrational; in other words, for all real numbers n.

We might expect the integration rule to hold for all real numbers also. But there is one snag: in working it out, we divided by $n+1$. Since division by zero does not make sense, the rule will not hold when $n+1 = 0$, that is, when $n = -1$. So we conclude that

Rule 3

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c$$

for all real numbers n, except $n = -1$.

When $n = -1$, $\int x^n dx$ becomes $\int x^{-1} dx = \int \frac{1}{x} dx$. We don't need to worry that the rule above doesn't apply in this case, because we already know the integral of $\frac{1}{x}$.

Since

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \qquad \text{we have} \qquad \int \frac{1}{x} dx = \ln x + c.$$

Examples

Find

i. $\int x^3 dx$

ii. $\displaystyle\int \frac{dx}{x^2}$

iii. $\int \sqrt{x} dx$

Solutions

i $\int x^3 dx = \dfrac{1}{(3+1)}x^4 + c = \dfrac{1}{4}x^4 + c.$ ← replacing n by 3 in the formula

ii $\int \dfrac{dx}{x^2} = \int x^{-2} dx = \dfrac{1}{-2+1}x^{-2+1} + c = -\dfrac{1}{x} + c.$ ← replacing n by -2 in the formula

iii $\int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \dfrac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + c = \dfrac{2}{3}x^{\frac{3}{2}} + c.$ ← replacing n by $\frac{1}{2}$

Exercises 3.1

1. Find anti-derivatives of the following functions:

i	x^5	**ii**	x^9
iii	x^{-4}	**iv**	$\frac{1}{x^2}$
v	$\frac{1}{\sqrt{x}}$	**vi**	$\sqrt[3]{x}$
vii	$x^{\sqrt{2}}$	**viii**	$x\sqrt{x}$
ix	$\frac{1}{x^\pi}$		

2. Find the following integrals:

i. $\int -3x dx$

ii. $\int \left(x^3 + 3x^2 + x + 4\right) dx$

iii. $\int \left(x - \dfrac{1}{x}\right) dx$

iv. $\int \left(x - \dfrac{1}{x}\right)^2 dx$ Hint: multiply out the expression

v. $\int \left(\dfrac{2}{\sqrt{x}} + \dfrac{\sqrt{x}}{2}\right) dx$

vi. $\int \dfrac{2x^4 + x^2}{x} dx$ Hint: divide through by the denominator

vii. $\int \left(\dfrac{3 + 5x - 6x^2 - 7x^3}{2x^2}\right) dx$ Hint: divide through by the denominator

At this stage it is very tempting to give a list of standard integrals. However, you are NOT encouraged to memorise integration formulae, but rather to become VERY familiar with the list of derivatives and to practise recognising a function as the derivative of another function.

If you try memorising *both* differentiation *and* integration formulae, you will one day mix them up and use the wrong one. And there is absolutely *no need* to memorise the integration formulae if you know the differentiation ones.

It is much better to recall the way in which an integral is defined as an anti-derivative. *Every time you perform an integration* you should pause for a moment and check it by differentiating the answer to see if you get back the function you began with. This is a very important habit to develop. There is no need to write down the checking process every time, often you will do it in your head, but if you get into this habit you will avoid a lot of mistakes.

Examples

Find the following indefinite integrals:

i. $\int \left(e^x + 3x^{\frac{5}{2}}\right) dx$

ii. $\int (5\csc^2 x + 3\sec^2 x) dx$

Solutions

i.

$$\begin{aligned} \int \left(e^x + 3x^{\frac{5}{2}}\right) dx &= \int e^x dx + 3\int x^{\frac{5}{2}} dx \\ &= e^x + 3 \cdot \frac{1}{\frac{5}{2}+1} x^{\frac{5}{2}+1} + c \\ &= e^x + 3 \cdot \frac{2}{7} x^{\frac{7}{2}} + c \\ &= e^x + \frac{6}{7} x^{\frac{7}{2}} + c. \end{aligned}$$

ii.

$$\begin{aligned} \int (5\csc^2 x + 3\sec^2 x) dx &= -5\int (-\csc^2 x) dx + 3\int \sec^2 x dx \\ &= -5\cot x + 3\tan x + c. \end{aligned}$$

Exercises 3.2

Integrate the following functions with respect to x:

i. $10e^x - 5\sin x$

ii. $\sqrt{x}(x^2 + x + 1)$ Hint: Multiply through by $\sqrt{x}$, and write with fractional exponents.

iii. $\dfrac{5}{\sqrt{(1-x^2)}} + \dfrac{1}{\sqrt{x}}$

iv. $\dfrac{x^3 + x + 1}{1 + x^2}$ Hint: Divide through by $1+x^2$, and consult table of derivatives.

v. $\dfrac{\tan x}{\sin x \cos x}$ Hint: Write $\tan x$ as $\frac{\sin x}{\cos x}$ and simplify.

vi. $\tan^2 x$ Hint: Remember the formula $1 + \tan^2 x = \sec^2 x$.

(If you are not familiar with inverse trig functions, omit parts **iii** and **iv**.)

Hint: In order to get some of the functions above into a form in which we can recognise what they are derivatives of, we may have to express them differently. Try to think of ways in which they could be changed that would be helpful.

4 Solutions to exercises

Exercises 1.1

i $\frac{d}{dx}(\sin x) = \cos x$, so $\int \cos x dx = \sin x + c.$

ii $\frac{d}{dx}(x^5) = 5x^4$, so $\int 5x^4 dx = x^5 + c.$

iii $\frac{d}{dx}(e^x) = e^x$, so $\int e^x dx = e^x + c.$

iv $\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}$, so $\int -\frac{2}{x^3} dx = \frac{1}{x^2} + c.$

$\left(\frac{1}{x^2} = x^{-2} \text{ and } \frac{d}{dx}(x^{-2}) = -2x^{-3} = -\frac{2}{x^3}\right)$

v $\frac{d}{dx}(x) = 1$, so $\int 1 dx = x + c.$

(**Note:** $\int 1dx$ is usually written as $\int dx$.)

vi $\frac{d}{dx}(\ln x) = \frac{1}{x}$, so $\int \frac{1}{x} dx = \ln x + c.$

Exercises 1.2

Note: All these answers can be checked by differentiating!

i $\int (-\sin x) dx = \cos x + c.$

ii $\int 3x^2 dx = x^3 + c.$

iii $\int 2dx = 2x + c.$

iv $\int \sec^2 x dx = \tan x + c.$

v $\int \frac{3}{2} x^{\frac{1}{2}} dx = x^{\frac{3}{2}} + c.$

vi $\int -\frac{1}{x^2} dx = \frac{1}{x} + c.$

Exercises 2

i $\int (\cos x + \sin x) dx = \sin x - \cos x + c.$

ii $\int (e^x - 1) dx = e^x - x + c.$

iii $\int (1 - 10x + 9x^2) dx = x - 5x^2 + 3x^3 + c.$

iv $\int (3\sec^2 x + \frac{4}{x}) dx = 3\tan x + 4\ln x + c.$

Exercises 3.1

1. **i** $\int x^5 dx = \frac{1}{6}x^6 + c.$

ii $\int x^9 dx = \frac{1}{10}x^{10} + c.$

iii $\int x^{-4} dx = -\frac{1}{3}x^{-4+1} = -\frac{1}{3}x^{-3} + c = -\frac{1}{3x^3} + c.$

iv $\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{1}{-1}x^{-1} + c = -\frac{1}{x} + c.$

v $\int \frac{1}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} dx = 2x^{\frac{1}{2}} + c = 2\sqrt{x} + c.$

vi $\int \sqrt[3]{x} dx = \int x^{\frac{1}{3}} dx = \frac{3}{4}x^{\frac{1}{3}+1} = \frac{3}{4}x^{\frac{4}{3}} + c.$

(**Note:** In exercises like **v** and **vi** above, it is easier to work out what power of x is required, and then to work out what coefficient is needed to give the correct answer on differentiating. This is usually better than substituting for n in $\frac{1}{n+1}x^{n+1}$. So **v** is more easily done by saying (mentally) " $-\frac{1}{2}+1=\frac{1}{2}$, so the answer will involve $x^{\frac{1}{2}}$. Now $2x^{\frac{1}{2}}$ will give a coefficient of 1 when differentiated so $\int x^{-\frac{1}{2}} dx = 2x^{\frac{1}{2}} + c$ ").

vii $\int x^{\sqrt{2}} dx = \frac{1}{\sqrt{2}+1}x^{\sqrt{2}+1} + c.$

viii $\int x\sqrt{x} dx = \int x^{\frac{3}{2}} dx = \frac{2}{5}x^{\frac{5}{2}} + c = \frac{2}{5}x^2\sqrt{x} + c.$

ix $\int \frac{1}{x^\pi} dx = \int x^{-\pi} dx = \frac{1}{-\pi+1}x^{-\pi+1} + c = -\frac{1}{(\pi-1)x^{\pi-1}} + c.$

2. **i** $\int -3x dx = -3 \cdot \frac{1}{2}x^2 + c = -\frac{3}{2}x^2 + c.$

ii $\int (x^3 + 3x^2 + x + 4) dx = \frac{1}{4}x^4 + x^3 + \frac{1}{2}x^2 + 4x + c.$

iii $\int (x - \frac{1}{x}) dx = \frac{1}{2}x^2 - \ln x + c.$

iv $\int (x - \frac{1}{x})^2 dx = \int (x^2 - 2 + \frac{1}{x^2}) dx$

$= \frac{1}{3}x^3 - 2x - \frac{1}{x} + c.$ (Recall, $\int \frac{1}{x^2} dx = -\frac{1}{x}$)

v $\int \left(\frac{2}{\sqrt{x}} + \frac{\sqrt{x}}{2}\right) dx = 2\int x^{-\frac{1}{2}} dx + \frac{1}{2}\int x^{\frac{1}{2}} dx$

$$= 2 \cdot 2x^{\frac{1}{2}} + \frac{1}{2} \cdot \frac{2}{3}x^{\frac{3}{2}} + c$$

$$= 4\sqrt{x} + \frac{1}{3}x\sqrt{x} + c.$$

vi $\int \frac{2x^4 + x^2}{x}dx = \int (2x^3 + x)dx$

$$= 2 \cdot \frac{1}{4}x^4 + \frac{1}{2}x^2 + c$$

$$= \frac{1}{2}x^4 + \frac{1}{2}x^2 + c.$$

vii $\int \frac{3 + 5x - 6x^2 - 7x^3}{2x^2}dx = \int \left(\frac{3}{2x^2} + \frac{5}{2x} - 3 - \frac{7x}{2}\right) dx$

$$= \frac{3}{2}\int x^{-2}dx + \frac{5}{2}\int \frac{1}{x}dx - 3\int dx - \frac{7}{2}\int xdx$$

$$= \frac{3}{2}(-\frac{1}{x}) + \frac{5}{2}\ln x - 3x - \frac{7}{2} \cdot \frac{1}{2}x^2 + c$$

$$= -\frac{3}{2x} + \frac{5}{2}\ln x - 3x - \frac{7}{4}x^2 + c.$$

Exercises 3.2

i $\int (10e^x - 5\sin x)dx = 10e^x + 5\cos x + c.$

ii $\int \sqrt{x}(x^2 + x + 1)dx = \int \left(x^{\frac{5}{2}} + x^{\frac{3}{2}} + x^{\frac{1}{2}}\right) dx$

$$= \frac{2}{7}x^{\frac{7}{2}} + \frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + c.$$

iii $\int \left(\frac{5}{\sqrt{(1 - x^2)}} + \frac{1}{\sqrt{x}}\right) dx = 5\sin^{-1} x + 2\sqrt{x} + c.$

iv $\int \frac{x^3 + x + 1}{1 + x^2}dx = \int \frac{x(x^2 + 1) + 1}{1 + x^2}dx$

$$= \int \left(x + \frac{1}{1 + x^2}\right) dx$$ Dividing through by $(1 + x^2)$

$$= \frac{1}{2}x^2 + \tan^{-1} x + c.$$

v $\int \frac{\tan x}{\sin x \cos x}dx = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\sin x \cos x}dx$ Writing $\tan x = \frac{\sin x}{\cos x}$

$$
\begin{aligned}
&= \int \frac{1}{\cos^2 x} dx \\
&= \int \sec^2 x dx \\
&= \tan x + c.
\end{aligned}
$$

vi $\int \tan^2 x dx$

$$
\begin{aligned}
&= \int (\sec^2 x - 1) dx \qquad \text{Using } \tan^2 x = \sec^2 x - 1 \\
&= \int \sec^2 x dx - \int 1 dx \\
&= \tan x - x + c.
\end{aligned}
$$