Mathematics Learning Centre

The University of Sydney

Integration: Another look at areas

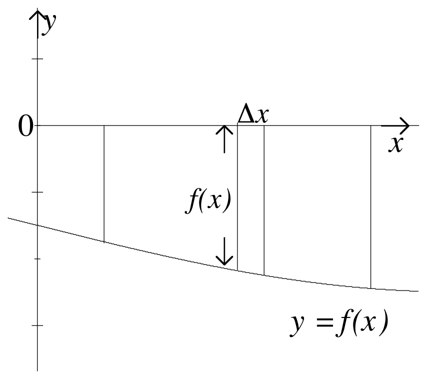
Mary Barnes

1 Another Look at Areas

The definite integral $\int_a^b f(x)dx$ is defined as the limit of a particular type of sum, without placing any restrictions on whether the function $f(x)$ is positive or negative.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \qquad (1)$$

We have seen that, if $f(x)$ is positive, $\int_a^b f(x)dx$ is equal to the area between the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$, (which we refer to as 'the area under the curve'). The natural question to ask now is: what does $\int_a^b f(x)dx$ equal if $f(x)$ is negative? Can we represent it as an area in this case too; perhaps 'the area above the curve'?
If we go back to the definition of $\int_a^b f(x)dx$ as the limit of a sum (see (1)), we can see clearly that if $f(x)$ is always negative then each of the terms $f(x_i)\Delta x$ will also be negative (since Δx is positive).



So the sum $\sum_{i=1}^{n} f(x_i)\Delta x$ will be a sum of negative terms and so will be negative too. And when we let n approach infinity and pass to the limit, that will be negative also.

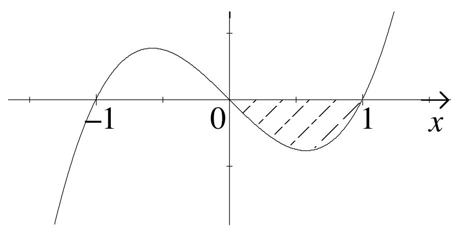
Thus, if $f(x)$ is negative for x between a and b, $\displaystyle\int_a^b f(x)dx$ will also be negative.

Now areas are, by definition, positive. If we ignore the fact that each of the terms $f(x)\Delta x$ is negative, and consider its **numerical value** only, we can see that it is numerically equal to the area of the rectangle. And, if we go through the usual process, adding up the areas of all the little rectangles and taking the limit, we find that $\displaystyle\int_a^b f(x)dx$ is **numerically equal** to the area between the curve and the x-axis.

So to find the area, we calculate $\displaystyle\int_a^b f(x)dx$, which will turn out to be negative, and then take its numerical (i.e. absolute) value.

To see this more clearly, let's look at an example. Consider the curve, $y = x(x^2 - 1)$. This is a cubic curve, and cuts the x-axis at -1, 0 and 1. A sketch of the curve is shown below. Let us find the shaded area. First we calculate the definite integral $\displaystyle\int_0^1 x(x^2 - 1)dx$.

$$\begin{aligned}\int_0^1 x(x^2-1)dx &= \int_0^1 (x^3 - x)dx \\ &= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_0^1 \\ &= \left(\frac{1}{4} - \frac{1}{2}\right) - (0 - 0) \\ &= -\frac{1}{4}.\end{aligned}$$



Since $x(x^2 - 1)$ is negative when x lies between 0 and 1, the definite integral is also negative, as expected. We can conclude that the area required is $\frac{1}{4}$ square units.

As a check, let us find the area of the other 'loop' of the curve, i.e. the area between the curve and the x-axis from -1 to 0. Since $x(x^2 - 1)$ is positive for this range of values of x, the area will be given by

$$\begin{aligned}\int_{-1}^{0} x(x^2-1)dx &= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{0} \\ &= (0-0) - \left(\frac{1}{4} - \frac{1}{2}\right) \\ &= \frac{1}{4}.\end{aligned}$$

This is the answer we would expect, since a glance at the diagram shows that the curve has 'point symmetry' about the origin. If we were to rotate the whole graph through 180°, the part of the curve to the left of the origin would fit exactly on top of the part to the right of the origin, and the unshaded loop would fit on top of the shaded loop. So the areas of the two loops are the same.

Now let us calculate $\int_{-1}^{1} x(x^2 - 1)dx$.

$$\begin{aligned}\int_{-1}^{1} x(x^2-1)dx &= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{1} \\ &= \left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right) \\ &= 0.\end{aligned}$$

This makes it very clear that

a definite integral does not always represent the area under a curve.

We have found that

1. If $f(x)$ is **positive** between a and b, then $\int_a^b f(x)dx$ does represent the area under the curve.

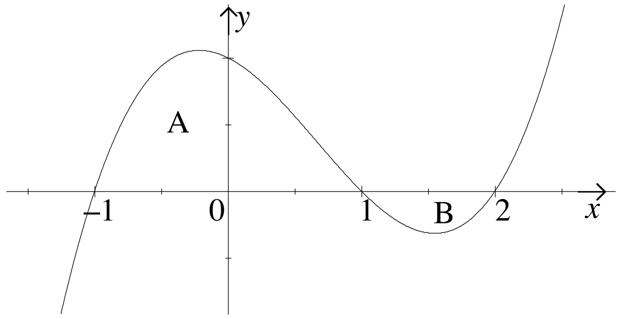
2. If $f(x)$ is **negative** between a and b, then $\left|\int_a^b f(x)dx\right|$ represents the area **above** the curve, since the value of $\int_a^b f(x)dx$ is negative.

3. If $f(x)$ is **sometimes positive and sometimes negative** between a and b, then $\int_a^b f(x)dx$ measures the **difference in area** between the part above the x-axis and the part below the x-axis. (In the example above, the two areas were equal, and so the difference came out to be zero.)

Let's look at another example.

Consider the function $y = (x + 1)(x - 1)(x - 2) = x^3 - 2x^2 - x + 2$.

This is a cubic function, and the graph crosses the x-axis at -1, 1 and 2. A sketch of the graph is shown.



The area marked A is given by

$$\begin{aligned}\int_{-1}^{1}(x^3 - 2x^2 - x + 2)dx &= \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-1}^{1} \\ &= \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right) - \left(\frac{1}{4} + \frac{2}{3} - \frac{1}{2} - 2\right) \\ &= -\frac{4}{3} + 4 = 2\frac{2}{3}.\end{aligned}$$

So the area of A is $2\frac{2}{3}$ square units.

The area marked B can be found by evaluating

$$\int_{1}^{2}\left(x^3 - 2x^2 - x + 2\right)dx.$$

This works out as $-\frac{5}{12}$. (The details of the calculation are left to you.)

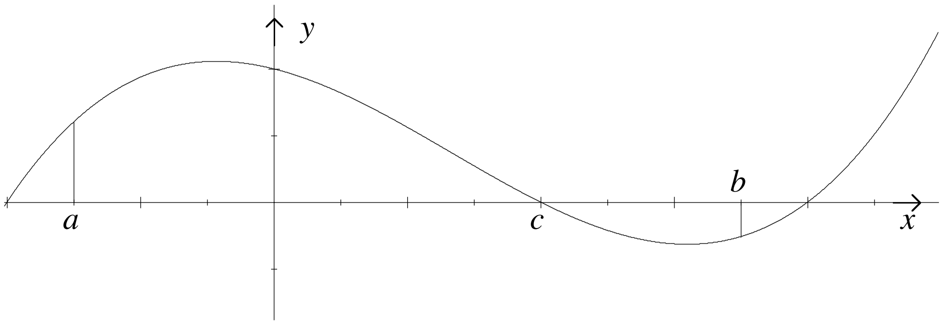
So the area of B is $\frac{5}{12}$ square units.

If we calculate $\int_{-1}^{2}(x^3 - 2x^2 - x + 2)dx$ the answer will be the difference between the area of A and the area of B, that is, $2\frac{1}{4}$ square units. (Check it out for yourself.)

If we want the total area enclosed between the curve and the x-axis we must **add** the area of A and the area of B.

i.e. $2\frac{2}{3} + \frac{5}{12} = 3\frac{1}{12}$ square units.

WARNING In working out area problems you should *always* sketch the curve first. If the function is sometimes positive and sometimes negative in the range you are interested in, it may be necessary to divide the area into two or more parts, as shown below.



The area between the curve and the x-axis from a to b is NOT equal to $\int_a^b f(x)dx$.

Instead, it is $\int_a^c f(x)dx + |\int_c^b f(x)dx|$.

Before you can calculate this, you must find the value of c, i.e. find the point where the curve $y = f(x)$ crosses the x-axis.

Exercises 1

1. Find the area enclosed by the graph of $y = 3x^2(x-4)$ and the x-axis.

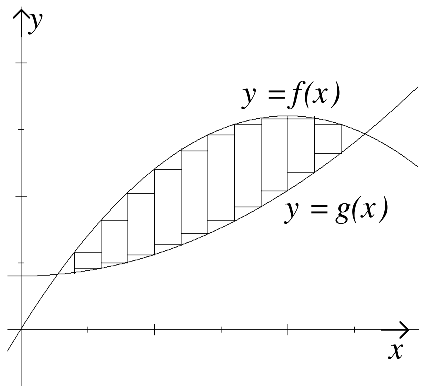
2. **i.** Find the value of $\int_0^{2\pi} \sin x dx$.

ii. Find the area enclosed between the graph of $y = \sin x$ and the x-axis from $x = 0$ to $x = 2\pi$.

3. Find the total area enclosed between the graph of $y = 12x(x+1)(2-x)$ and the x-axis.

2 The Area Between Two Curves

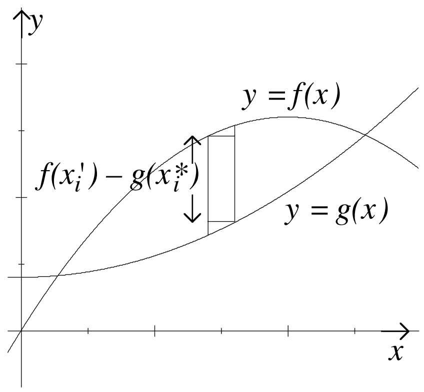
Sometimes we want to find, not the area between a curve and the x-axis, but the area enclosed between two curves, say between $y = f(x)$ and $y = g(x)$.
We can approach this problem in the same way as before by dividing the area up into strips and approximating the area of each strip by a rectangle. The lower sum is found by calculating the area of the interior rectangles as shown in the diagram.



The height of each interior rectangle is equal to the difference between the least value of $f(x)$, $f(x')$, and the greatest value of $g(x)$, $g(x^*)$, in the rectangle. The area of the ith rectangle is $(f(x_i') - g(x_i^*))\Delta x$.

$$\text{The lower sum} = \sum_{i=1}^{n}(f(x_i') - g(x_i^*))\Delta x.$$

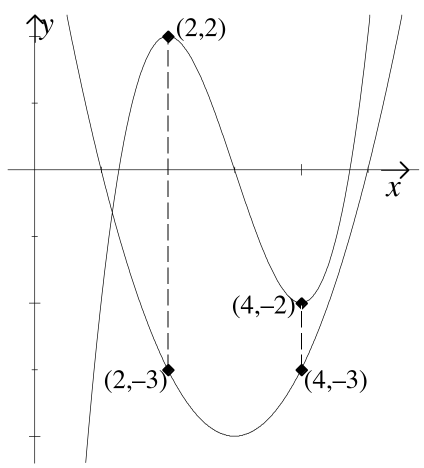
The upper sum can be found in the same way. The area enclosed between the curves is sandwiched between the lower sum and the upper sum.



When we pass to the limit as $\Delta x \to 0$, we get

$$\text{Area enclosed between the curves} = \int_a^b (f(x) - g(x))dx.$$

Note that the height is *always* $f(x) - g(x)$, even when one or both of the curves lie below the x-axis.
For example, if for some value of x, $f(x) = 2$ and $g(x) = -3$, the distance between the curves is $f(x) - g(x) = 2 - (-3) = 5$, or, if $f(x) = -2$ and $g(x) = -3$, the distance between the curves is $(-2) - (-3) = 1$ (see the diagram).



So, to find the area enclosed between two curves, we must:

1. Find where the curves intersect.

2. Find which is the upper curve in the region we are interested in.

3. Integrate the function (upper curve − lower curve) between the appropriate limits.

In other words, if two curves $f(x)$ and $g(x)$ intersect at $x = a$ and $x = b$, and $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\text{Area enclosed between the curves} = \int_a^b (f(x) - g(x))dx.$$

Exercises 2

(Remember to draw a diagram first, before beginning any problem.)

1. Find the area enclosed between the parabola $y = x(x-2)$ and the line $y = -x + 2$.

2. Find the area enclosed between the two parabolas $y = x^2 - 4x + 2$ and $y = 2 - x^2$.

3. Check that the curves $y = \sin x$ and $y = \cos x$ intersect at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$, and find the area enclosed by the curves between these two point.

4. **i.** Sketch the graphs of the function $y = 6 - x - x^2$ and $y = x^3 - 7x + 6$.

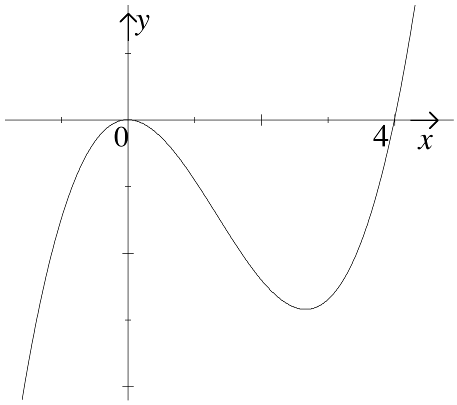
ii. Find the points of intersection of the curves.

iii. Find the total area enclosed between them.

3 Solutions to exercises

Exercises 1

1. First, draw a graph.
The area is below the x-axis, so we first calculate $\int_0^4 3x^2(x-4)dx$.

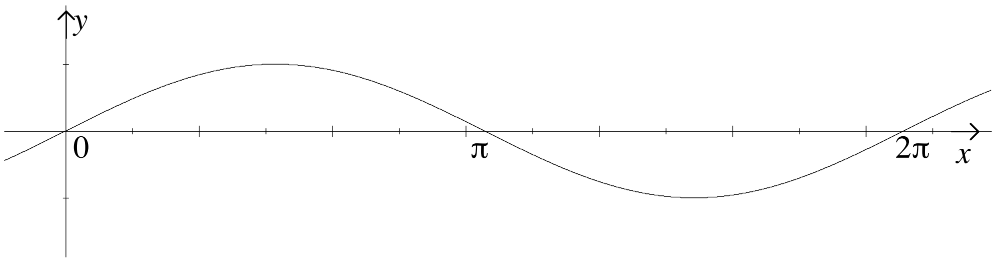


$$\begin{aligned}\int_0^4 3x^2(x-4)dx &= \int_0^4 (3x^3-12x^2)dx \\ &= \left[\frac{3}{4}x^4 - 4x^3\right]_0^4 \\ &= -64.\end{aligned}$$

The required area is therefore 64 units.

2. **i.** $\int_0^{2\pi} \sin x dx = [-\cos x]_0^{2\pi} = -\cos 2\pi + \cos 0 = -1+1 = 0$

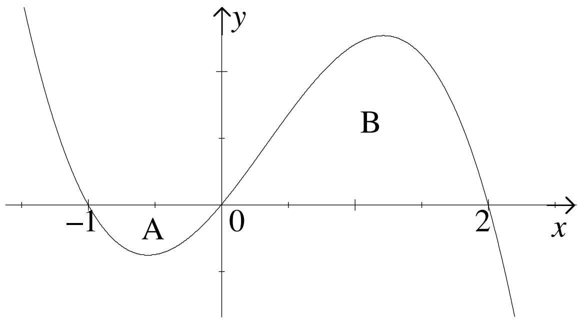
ii.



From the graph we see that the area

$$\begin{aligned}\text{Area} &= \int_0^{\pi} \sin x dx + |\int_{\pi}^{2\pi} \sin x dx| \\ &= [-\cos x]_0^{\pi} + |[-\cos x]_{\pi}^{2\pi}| \\ &= (-\cos\pi + \cos 0) + |-\cos 2\pi + \cos\pi| \\ &= (-(-1)+1) + |-1+(-1)| \\ &= 4.\end{aligned}$$

3. The graph of the curve cuts the x-axis at -1, 0 and 2.
The total area = area A + area B.



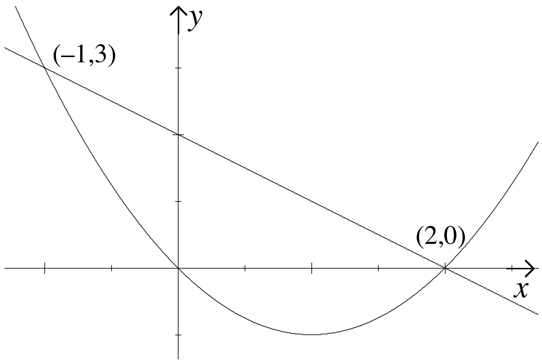
$$\begin{aligned}\text{Area A} &= |\int_{-1}^{0} 12x(x+1)(2-x)dx| \\ &= |\int_{-1}^{0} (-12x^3+12x^2+24x)dx| \\ &= |\left[-3^4 + 4x^3 + 12x\right]_{-1}^{0}| \\ &= |0-(-3-4+12)| \\ &= |-5| = 5.\end{aligned}$$

$$\text{Area B} = \int_0^2 12x(x+1)(2-x)dx$$
$$= \left[-3x^4 + 4x^3 + 12x^2\right]_0^2 = (-48+32+48) = 32.$$

Therefore the total area is $5 + 32 = 37$ square units.

Exercises 2

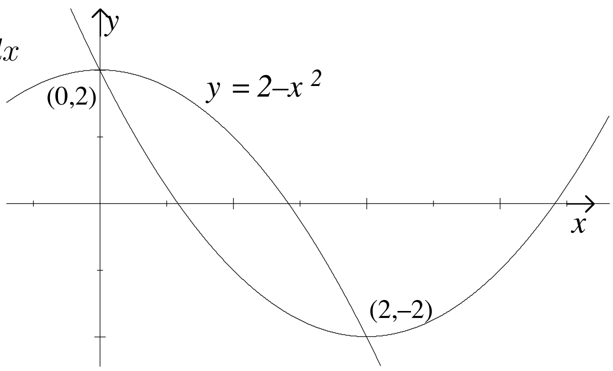
1. The curves $y = x^2 - 2x$ and $y = -x + 2$ intersect where $x^2 - 2x = -x + 2$. i.e. at $x = -1$ or $x = 2$.
The upper curve is $y = -x + 2$.



$$\begin{aligned}
\text{Area} &= \int_{-1}^{2}((-x+2) - (x^2-2x))dx \\
&= \int_{-1}^{2}(2 + x - x^2)dx \\
&= \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{2} \\
&= (4 + 2 - \frac{8}{3}) - (-2 + \frac{1}{2} + \frac{1}{3}) \\
&= 4\frac{1}{2}.
\end{aligned}$$

2. The curves intersect where $x^2 - 4x + 2 = 2 - x^2$ i.e. $2x^2 - 4x = 0$ i.e. $x = 0$ or $x = 2$.
The upper curve is $y = 2 - x^2$ (see sketch).

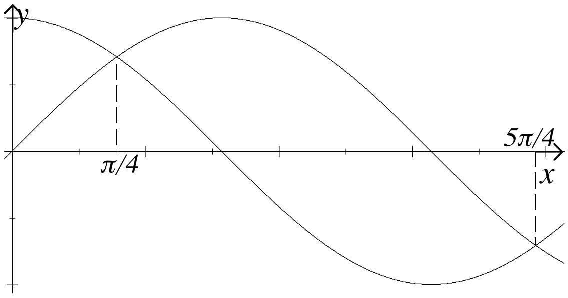
$$\begin{aligned}
\text{Area} &= \int_{0}^{2}((2-x^2) - (x^2-4x+2))dx \\
&= \int_{0}^{2}(4x - 2x^2)dx \\
&= \left[2x^2 + \frac{2}{3}x^3\right]_{0}^{2} \\
&= (8 - \frac{2}{3}\cdot 8) - 0 \\
&= 2\frac{2}{3}.
\end{aligned}$$



3. When $x = \frac{\pi}{4}$, $\sin x = \frac{1}{\sqrt{2}}$ and $\cos x = \frac{1}{\sqrt{2}}$.

When $x = \frac{5\pi}{4}$, $\sin x = -\frac{1}{\sqrt{2}}$ and $\cos x = -\frac{1}{\sqrt{2}}$.

So the curves $y = \sin x$ and $y = \cos x$ intersect at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.



$$\begin{aligned}
\text{Area} &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)dx \\
&= [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\
&= (-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4}) + (\cos\frac{\pi}{4} + \sin\frac{\pi}{4}) \\
&= \frac{4}{\sqrt{2}} \\
&= 2\sqrt{2}.
\end{aligned}$$

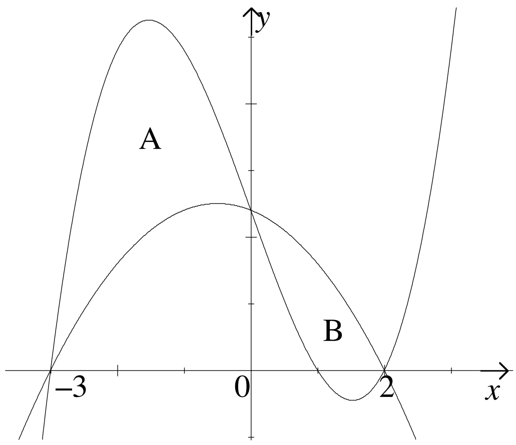
4. (i) and (ii) The curves are easier to sketch if we first find the points of intersection: they meet where $x^3 - 7x + 6 = 6 - x - x^2$.
That is,

$$x^3 + x^2 - 6x = 0$$

or

$$x(x-2)(x+3) = 0.$$

So the points of intersection are $(0,6)$; $(2,0)$; and $(-3,0)$.
The first curve is an 'upside-down' parabola, and the second a cubic.
Total area = area A + area B.



$$\begin{aligned}
\text{Area A} &= \int_{-3}^{0} ((x^3 - 7x + 6) - (6 - x - x^2))dx \\
&= \int_{-3}^{0} (x^3 + x^2 - 6x)dx \\
&= \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2\right]_{-3}^{0} \\
&= 15\frac{3}{4}.
\end{aligned}$$

$$\begin{aligned}
\text{Area B} &= \int_{0}^{2} ((6 - x - x^2) - (x^3 - 7x + 6))dx \\
&= \int_{0}^{2} (6x - x^2 - x^3)dx \\
&= 5\frac{1}{3}.
\end{aligned}$$

$\therefore$ the total area $= 15\frac{3}{4} + 5\frac{1}{3} = 21\frac{1}{12}$ square units.